Rules for Rubik's Family Cubes of All Sizes

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Summary

Movement limitations in Rubik's family cubes give rise to a set of rules which define restraints on what is possible. While much has been written about the rules that apply to the standard Rubik's cube, much less is available for cubes of larger size. This document presents and justifies rules which in most cases need to be expressed as functions of cube size. The attributes considered are mostly mathematical in nature. Particular emphasis is given to parity and the number of unreachable states for cubes of any size, as there is a dearth of information available on these topics other than for the standard Rubik's cube.

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1. Definitions

Cube size	The standard Rubik's cube is often referred to as a 3x3x3 cube. That cube will be referred to as a size 3 cube and in general an n x n x n cube will be referred to as a size n cube.
Rubik cube family	Cubes that have similar rotational properties to the standard Rubik's size 3 cube and obey generalized rules for a size n cube are considered to be members of the Rubik cube family. Cubes of size 2 and above that meet this condition are available.
Hardware cube	A hardware or physical cube is a Rubik's family cube that comes as a single-size hand-held object.
Software cube	A software cube is a program that emulates and presents the cube in some form on a computer monitor and allows the user to rearrange it. Software cubes that accommodate a range of cube sizes are available. Such cubes are not subject to the physical restraints that impose a size limit on the hardware forms.
Rule	One of a set of generalized laws that defines what is and what is not possible (usually in mathematical terms) for Rubik's family cubes.
Cubie	Individual cube elements will be referred to as cubies (others sometimes refer to them as "cubelets"). There are three types of cubies: corner cubies (three coloured surfaces), edge cubies (two coloured surfaces) and centre cubies (one coloured surface). The absolute centre cubies for odd size cubes sit on the central axes of the six faces and their relative positions never change.
Cubicle	A cubicle is the compartment in which a cubie resides. For a permutation, cubicles are considered to occupy fixed positions in the space occupied by the cube object but their contents (cubies) may shift position.
Facelet	A facelet is a visible coloured surface of a cubie (corner cubies have three facelets, edge cubies have two and centre cubies have one).
Cube state	A particular arrangement of the cubies will be referred to as a cube state. What looks the same is considered to be the same (unless specific mention to the contrary is made). Each state has equal probability of being produced after a genuine random scrambling sequence. A rotation of the whole cube does not change the state considered herein. In other texts the various states are often referred to as permutations or arrangements.
Cube layer	A cube layer is a one cubie width slice of the cube perpendicular to its axis of rotation. Outer layers (faces) contain more cubies than inner layers. For a cube of size n there will be n layers along any given axis.
Cube face	The meaning of a cube face depends on the context in which it is used. It usually means one of the six three-dimensional outer layers but can also refer to just the outside layer's surface which is perpendicular to its axis of rotation. The faces are usually designated as up (U), down (D), front (F), back (B), left (L) and right (R).
Cube style	Two cube styles are referred to in this document: firstly a standard cube with unmarked centres and secondly a cube with marked centres.

Set state	The set (or solved) state of a cube with unmarked centres is one for which a uniform colour appears on each of the six faces. For cubes with marked centres the set state is characterised by a unique arrangement of all centre cubies.
Scrambled state	The scrambled state is the starting point for unscrambling the cube. It arises when a cube in the set or any other state is subject to a large number of randomly chosen layer rotations.
Orbit	For a basic quarter turn of a cube layer for cubes of all sizes, sets-of-four cubies move in separate four-cubicle trajectories. When all the possible trajectories for a given cubie type are considered for the whole cube we will refer to all the possible movement positions as being in a given orbit. We consider that the size 3 cube has two orbits, one in which the eight corner cubies are constrained to move and one in which the 12 edge cubies are constrained to move. Transfer of cubies between these orbits is impossible. For cubes of size 4 and above we will also define an edge cubie orbit as comprising 12 cubies but will use the term *complementary orbit* to describe a pair of orbits between which edge cubies can move. A pair of complementary edge cubie orbits contains a total of 24 cubies. Cubes of size 4 and above include centre cubie orbits that contain 24 cubies. Transfer of cubies between one such orbit and another is not possible (applies to cubes of size 5 and above).
Move	A move is a quarter turn rotation of a layer or a sequence of such quarter turns that a person would apply as a single step.
Move notation	A clockwise quarter turn of an outer layer is usually expressed as U, D, F, B, L or R. In other respects the notation used varies among authors.
Algorithm	An algorithm defines a sequence of layer rotations to transform a given state to another (usually less scrambled) state. Usually an algorithm is expressed as a printable character sequence according to some move notation. An algorithm can be considered to be a “smart” move. All algorithms are moves but few moves are considered to be algorithms.
Permutation	A permutation of the cube as used herein means the act of permuting (i.e. rearranging) the positions of cubies. A permutation is an all-inclusive term which includes a sequence of quarter turns of any length. Even the solving of the cube from a scrambled state represents a permutation. The term "permutation" is used extensively by mathematicians who use Group Theory to quantify the process involved in a rearrangement of cubies. The term "permutation" is also often used to mean the state of the cube that results after it is rearranged but that meaning will not be used herein. In such cases the term “cube state” will be used. That allows the term “permutation” to be used when the permutation results in no change of state – an area of special interest for Rubik’s family cube permutations.
Parity	A cube permutation can be represented by a number of swaps of two cubies. If that number is even the permutation has even parity, and if the number is odd the permutation has odd parity.

2. Introduction

Rules for the standard Rubik's cube are used to define cube arrangements that are and are not reachable. Such rules are usually described in mathematical terms as numerical or logical constants. For instance the possible arrangement of cubies can be defined as an integer constant. In mathematical terms, whether a number is even, or whether a state is reachable, can be considered as a logical entity (having only a true or false value). When the Rubik's family cubes are considered as a group, most rules need to be defined as mathematical functions of the cube size.

Where applicable, results from the author's other documents are used and those documents need to be referred to for detailed justification for those results. Except for some special cases to illustrate specific issues, this document does not provide instructions on moves (algorithms) for use in cube solving. On-line help for the author's Unravel program software[1] and supporting documents[2][3] provide instructions and algorithms sufficient for solving cubes of various sizes with standard unmarked centres and marked centres respectively. There are many ways cubes can be solved and these references detail just one way.

3. Number of Cubies and Facelets

For a cube of size n:

Number of corner cubies	=	8
Number of edge cubies	=	$12(n - 2)$
Number of centre cubies	=	$6(n - 2)^2$
Number of facelets	=	$6n^2$
Total number of cubies	=	$6(n - 1)^2 + 2$
Increase in total number of cubies for unit increase in cube size from n to $n + 1$	=	$12n - 6$

4. Cube Rotation Rules

All rules that define what cube states are reachable are a consequence of the cube's rotational movement possibilities.

There are three mutually perpendicular axes[4] of rotation for the cube. One set of axes defined in terms of the D, U, B, F, L and R faces can be considered to have a fixed orientation in space. Think of these axes as belonging to a cube-shaped container in which the cube object can be positioned in any of 24 orientations. One axis can be drawn through the centres of the D and U faces (the DU axis). The others are the BF and LR axes.

Another set of axes, can be defined for the cube object itself. These axes relate to the face colours, the most common being white, red, orange, yellow, green and blue. The axes are usually white-blue, red-orange and yellow-green. For odd size cubes these axes are always fixed relative to the internal frame of the cube object. For even size cubes these axes can be positioned in any of 24 orientations but remain fixed relative to the internal frame of the cube object after initial selections. The origin for the axes is the centre of the cube object.

The only way that cube state can be changed is by the rotation of cube layers about their axes of rotation. All changes of state involve rotation steps that can be considered as a sequence of single layer quarter turns.

For a basic quarter turn of a cube layer for cubes of all sizes, sets-of-four cubies move in separate four-cubicle trajectories. For the size 3 cube there will be two such trajectories on any given face (one for corner cubies and one for edge cubies) and corresponding trajectories on the other five faces. When considering the cube as a whole, the term orbit is used to include all such trajectories. The position of the absolute centre cubie for odd size cubes does not change when its face is rotated. For odd size cubes of size greater than 3 the central edge cubies are constrained to a single orbit as for the size 3 cube. In all other cases there will be complementary edge cubie orbits between which cubie movement is possible. No movement of edge cubies between non-complementary orbits is possible.

For odd size cubes the absolute centre cubies can be considered to reside on a fixed frame within the cube object. There can be no relative positional movement between the absolute centre cubies and they can be considered to form a fixed reference frame within the cube object. Rotation of these cubies about their own axes is possible but that rotation is relevant only when considering cubes with marked centre cubies. All cubie movements in odd size cubes can be considered as rotations relative to the fixed reference frame. For even size cubes no such reference frame can be observed from the external surfaces of the cube. Except for the absolute centre cubie for odd size cubes, the cubies move in separate 24 cubicle orbits. Movement of cubies between these separate orbits is not possible.

Since movement between orbits is not possible for centre cubies, separate markings for different orbits are not required when marked centre cubies are considered. If that were not the case, the six colours of the standard cube would need to be replaced with 24 separately identifiable markings. A simple 1-2-3-4 marking superimposed on the six face colours is the approach that has been used where possible in the Java version of the *Unravel* program[1][3]. A corner marking graphic has recently been added to allow centre cubie marking to be extended beyond the numerical marking limit.

5. Permutation and Orientation Rules

The rotation rules give rise to both cubie position and orientation limitations. Cubie position limitations are usually defined in terms of permutation parity rules.

5.1 Permutation Rules

A permutation of the cube, as defined in this document, means the act of permuting (i.e. rearranging) the state of the cube. Under this definition a cube move or rotation sequence is a permutation. Even the solving of the cube from a scrambled state represents a permutation. The term "permutation" is used extensively by mathematicians to quantify the process involved in a rearrangement of cubies. The term "permutation" may also be used to mean the state of the cube that results after it is rearranged but *cube state* in lieu of that meaning will be used herein. Permutations that result in no change of state can be considered as special cases that fit within the definition.

The relationship between the cube state after a move with that before a move can be expressed mathematically using Group Theory[5][6][7] to quantify permutations. Since every move can be considered as a sequence of quarter turn rotations, it is appropriate to examine what is involved in a quarter turn rotation. Except for the absolute centre cubie for odd size cubes, the cubies move in separate four-cubicle trajectories (also referred to as a 4-cycle movement since four quarter turns will restore the cubies in the specified trajectory to their original positions). A quarter turn of a 4-cubie set can be represented by three swaps as indicated below where swap 1-2 means the contents of cubicle 1 is swapped with the contents of cubicle 2, etc. A clockwise quarter turn example is considered.

Start	
1/A	2/B
4/D	3/C

Swap 1-2	
B	A
D	C

Swap 1-3	
C	A
D	B

Swap 1-4	
D	A
C	B

The parity of a permutation can be expressed in various ways:

Descriptive	Logical	Numerical
even	true	0
odd	false	1

The numerical form has the advantage of providing a simple means of expressing parity in mathematical terms. Consider the application of permutation p_1 which involves x_1 swaps followed by another permutation p_2 which involves x_2 swaps. The overall parity of p_1p_2 can then be expressed in modulo[2] form as $(x_1 + x_2)(\text{mod } 2)$.

Since a quarter turn is made up of a number of 4-cycles each involving three swaps, if the number of 4-cycles is odd, overall parity of the quarter turn permutation will be odd and vice versa.

Quarter turn permutation parity for a size *n* cube is given in the following table.

Cube size (odd or even)	Layer type	Number of 4-cycle movements	Overall parity
odd	inner	$n - 1$	even
odd	outer	$((n - 2)^2 - 1)/4 + (n - 1)$	even*
even	inner	$n - 1$	odd
even	outer	$(n/2 - 1)^2 + (n - 1)$	even if $n/2$ is even ** odd if $n/2$ is odd

* Since $((n - 2)^2 - 1)$ equals $(n - 1)(n - 3)$, a product of two consecutive even numbers, which must always be evenly divisible by 8.

** Since $(n/2 - 1)^2$ will be odd if $(n/2 - 1)$ is odd (i.e. if $n/2$ is even) giving overall even since $(n - 1)$ is odd. The reverse applies if $n/2$ is odd.

Summarising the above parity results we conclude:

- All permutations for odd size cubes have even overall parity.
- All individual quarter turns for even size cubes, where half the cube size is an odd number, have odd overall parity.
- For even size cubes where half the cube size is an even number, inner layer quarter turns have odd overall parity and outer layer quarter turns have even overall parity.

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[2] If *h* and *n* are positive integers, *h* modulo *n* (abbreviated to "*h* mod *n*") is the remainder that results if *h* is divided by *n*.

The above analysis considered the parity for corner (where applicable), edge and centre cubies combined. It is possible to consider these in isolation and when that is done an even combined quarter turn parity will involve a number of odd parity elements.

The parity rules as defined in the above table apply irrespective of how individual cubies are identified (e.g. whether or not all centre cubies in a given orbit have individually identifying markings). For normal cubes of size 4 and above, exchange of a centre cubie with any of the remaining centre cubies of the same face identification will result in exactly the same observed state for the cube. That makes it difficult to observe how compliance with parity rules is maintained and gives rise to the use of terminology such as "parity error correction". For the central edge cubie for odd size cubes the behaviour is the same as that for the size 3 cube. Only half the conceivable orientations are reachable.

For the edge cubie sets, comprising 12 complementary pairs (24 cubies total), there is no restriction on position as for the central edge cubies for odd size cubes. However, for any given position, only one of the two conceivable orientations is reachable.

For cubes with marked centres there are 4! (equal to 24) possible arrangements for the four cubies in a given orbit for the first five faces but that is halved[3] (equal to 12) for the last face. For cubes of odd size this result can be inferred from the even parity rule for all permutations. For cubes of even size the same result applies but in that case the rule needs to be generalized to: "Any permutation that results only in a rearrangement of centre cubies in a given orbit must have even parity".

5.2 Identity Permutations

As indicated above, some cubie permutations for even size cubes may have odd parity. However, a permutation that results in no change of state, which is often referred to as an *Identity Permutation*, must always have even parity.

For example, if a permutation p_1 is used to scramble the cube from the set state and permutation p_2 is used to solve the cube from the scrambled state, then permutation p_1p_2 must have even parity. If, for an even size cube, p_1 is even then p_2 must also be even, and if p_1 is odd then p_2 must be odd.

A fundamental property[6] of the standard Rubik's size 3 cube is that any permutation applied a sufficient number of times will result in the cube state returning to that which applied before the first application of the permutation. That property also applies to Rubik's family cubes of any size. The permutation (move) cycle length is the minimum number of times the permutation needs to be applied for the new state to correspond with the initial state. The cycle length is also referred to as the *order* of the permutation. The overall permutation comprising the defined permutation repeated the cycle length times represents an Identity Permutation.

The Java version of the Unravel program[1] has an option which allows users to determine the cycle length of user-defined permutations for cubes of user-specified size..

For any given permutation, a cube of any size greater than 3, which is subject to only outer layer rotations, will return the same cycle length. For any given permutation, cycle length may vary according to other variables as indicated in the following table.

Variable	Effect
Cube size	Cube size can have a major effect on cycle length.
Initial cube state*	For cubes with unmarked centres, the cycle length for a cube with an initial set state may be different to that for an initial scrambled state. This arises because centre cubies can end up in different positions that appear identical. For cubes with marked centres, the cycle length is independent of initial state.
Cube style (unmarked or marked centres)	Cycle count for cubes with unmarked centres can be the same or lower than that for a cube with marked centres.
Spatial orientation	The state of a cube is not changed if its spatial orientation is changed (e.g. if a hardware cube is turned upside down). There are 24 ways a cube can be presented spatially and all 24 need to be checked if correspondence with any spatial orientation of the initial state is to terminate the cycle count.

* For example the Unravel program permutation `#7B#3U-#7BU#7B-#3U-#7BUU` swaps a centre cubie on the `F` face with one in the equivalent position on the `U` face (where `#3U` etc. means the third row from the top is rotated a quarter turn clockwise about the `U` axis and "–"refers to anticlockwise). When applied to a set cube with unmarked centres it exhibits a cycle length of 4. When a random initial state is chosen for the unmarked cube or any initial state for a marked cube, the cycle length changes to 12.

Cycle lengths are more often even numbers but can also be odd. Some examples that illustrate the effect of different settings for variables (defined in the above table) are given in the following table. Marking does not apply for the size 2 cube. In general smart moves (algorithms) usually have short cycle lengths. Cycle lengths and count times can be very large.

Cube size	Marking	Initial state	Final orientation	Permutation*	Permutation parity	Cycle length
All	Either		Not relevant	`F`	Odd if *n* is even and *n*/2 is odd	4
All	Either	Set	Initial	`WF`	Even	4
All	Either	Set	Any	`WF`	Even	1
All	Unmarked	Set	Initial	`FR`	Even	105
All	Unmarked	Scrambled	Initial	`FR`	Even	420
All	Marked	Set	Initial	`FR`	Even	420
All	Either	Set	Initial	`FUR`	Odd if *n* is even and *n*/2 is odd	84
All	Unmarked	Set	Initial	`LLU-FBLL-BFULL`	Even	3
All	Marked	Set	Initial	`LLU-FBLL-BFULL`	Even	6
All	Unmarked	Set	Initial	`-RDRFD-F`	Even	45
4+	Unmarked	Scrambled	Initial	`-RDRFD-F`	Even	90
All	Marked	Set	Initial	`-RDRFD-F`	Even	90
4	Either	Set	Initial	`2F2R`	Even	15
16	Unmarked	Set	Initial	`2F2R`	Even	105
16	Marked	Set	Initial	`2F2R`	Even	420
4	Either	Set	Initial	`2F3R`	Odd	6840
16	Unmarked	Set	Initial	`7F11R`	Even	75240
16	Marked	Set	Initial	`7F11R`	Even	150480
64	Unmarked	Set	Initial	`7F11R`	Even	1680
32	Either	Set	Initial	`FURBDL13F29R19B`	Even	526680
32	Either	Set	Initial	`FURBDL12F28R18B`	Odd	8953560

* -R etc. means a counter-clockwise outer layer quarter turn about the right face.

3R etc. means a clockwise quarter turn of the three outer-most layers about the R face axis.

WF means a clockwise quarter turn of the whole cube about the front face.

Some rules that apply to identity permutations that may or may not be illustrated in the above short sample are:

- If permutation parity is odd cycle length is always even.
- Cycle length for any permutation involving only outer layer rotations is independent of cube size (but may vary with initial state or marking).
- Cycle lengths for permutations involving only inner layer rotations are always even.
- Cycle lengths for permutation involving only inner layer rotations may vary with cube size (noting that some permutations that are valid for larger cubes will not be valid for smaller ones).

5.3 Orientation Rules

Corner cubies have three possible orientations. Seven of the eight corner cubies can be arbitrarily located. Once the orientation of seven corner cubies is defined there is only one possible orientation for the last one.

Edge cubies for the size 3 cube and the central edge cubies for larger odd size cubes behave similarly. Eleven of the twelve edge cubes can be flipped independently, with the flip state of the last depending on the preceding ones.

For cubies of size 4 and above, movement of edge cubies within complementary orbits, containing a combined total of 24 edge cubies, is possible. All edge cubies in the set of 24 can be arbitrarily placed if some centre cubie movement is permitted. Orientation cannot be changed independently of placement.

Orientation of centre cubies is relevant if the six absolute centre cubies for odd size cubes have markings to indicate rotational status. For such cubes the rotational status of five centre cubies can be arbitrarily set but the only reachable states for the last one is the current state and a half rotation from that state.

For cubes of size 4 and above the sets-of-four centre cubies on any face behave like the centre cubies for the size 3 cube (i.e. if no change to the cube arrangement other than to the set-of-four cubies under consideration is permitted, then the only .possibilities are the current arrangement and a half turn from the current arrangement). However, for normal cubes this condition can be met by changes that are not readily observable.

For the size 4 cube there is a single orbit of 24 centre cubies and for cubes of size greater than 4, there will be multiple 24 centre cubie orbits. For hardware cubes, the orientation of the centre cubies changes with position as the face is rotated. The orientation changes are visible only in cubes with marked centres[3]. For hardware cubes, if the position is known the orientation is known and vice versa. For software cubes it follows that restricting movement to only position changes meets all necessary rule requirements.

6. Reachable and Unreachable States

If a cube has at some previous time occupied the set state, then any state that can arise after legal moves is considered to be a reachable state. For small size cubes (size 2, 3 or 4) an unreachable state is one that cannot be reached by legal moves. For larger cubes there needs to be some further qualification on what is meant by an unreachable state. In this document we exclude notional movement between 24-cubie orbits for edge and for centre cubies.

6.1 General Relationship Between Reachable and Unreachable States

If, for a cube of any size, m represents the number of reachable states, u represents the number of unreachable states and t equals their sum:

$$t = u + m$$

$$\mathrm{t} = km \text{ where } k \text{ is a positive integer}$$

$$u = (k - 1)m$$

Both m and k are functions of cube size n. Values for m and k will be considered in the following sections.

6.2 Reachable States for Cubes of All Sizes

The number of reachable states is based on:

- Standard *permutation mathematics*.[9]
- Reduction factors that must be applied to reflect movement restrictions specific to Rubik's family cubes.

The number of reachable states for cubes of all sizes can be simply related to the numbers that are applicable to the size 3 and size 4 cubes. My reference[9] provides a derivation and justification for the general relationships for:

- a standard size n cube, and

- a size n cube for which all centre cubies in each 24-cubie orbit have identifying markings such that each one can have only one correct location for the solved state.

The results are reproduced here.

The standard unmarked cube can be considered to form a batch of special cases of the marked cube for which the set state represents a unique arrangement of all cubies and their orientations.

For cubes with unmarked centre cubies the following positive integer constants (represented by P, Q, R and S) apply.

Corner cubie possibilities for even size cubes	P	$(7!)\ 3^6$	$3.67416000000000\text{x } 10^6$
Central edge cubie possibilities for odd size cubes, multiplied by 24	Q	$24\ (12!)\ 2^{10}$	$1.17719433216000\text{x } 10^{13}$
Edge cubie possibilities for each complementary set (12 pairs)	R	$24!$	$6.20448401733239\text{x } 10^{23}$
Centre cubie possibilities for each quadruple set (6 groups of 4)	S	$(24!)/(4!)^6$	$3.24667053711000\text{x } 10^{15}$

Note: ! is the factorial symbol (N! means the product 1 × 2 × ... × N).

The number of reachable states m for a size n cube can be defined in terms of the factors P, Q, R and S.

$$m = \mathrm{P}\ \mathrm{Q}^{a}\ \mathrm{R}^{b}\ \mathrm{S}^{c}$$

where a, b and c are positive integer variables (functions of cube size n) as given below.

$$a = n \bmod 2 \text{ (i.e. 0 if } n \text{ is even or 1 if } n \text{ is odd)}$$
$$b = (n - 2 - a)/\ 2$$
$$c = ((n - 2)^2 - a)/\ 4$$

For even size cubes $\mathrm{Q}^a = 1$.

The value of S warrants further explanation. For a marked cube, for any specific arrangement of edge cubies, only half the conceivable states are reachable. Hence in that case the number of reachable states is 24! / 2. If the special markings are now removed, we need to reduce the above number of possibilities by a factor of 4! for each set of four identical centre cubies, except the 4! factor for the last set of four which is halved to account for unreachable states similar to that which applied for marked centre cubies. Hence the net number S of possible arrangements for the centre cubies of a size 4 cube becomes $\mathrm{S} = (24!)\ /\ (4!)^{6}$

For cubes with marked centres the values for P and R will be the same as above but those for Q and S will be different. For parameter Q we need to take account of the orientation of the absolute centre cubie for cubes of odd size. For parameter S we need to have identifying marking of all 24 centre cubies in each orbit (but the same markings can be used for each orbit). Define m_{M}, $\mathrm{Q_M}$ and $\mathrm{S_M}$ to be the changed parameters.

$$\mathrm{Q_M} = \mathrm{T\ Q} \quad \text{where } \mathrm{T} = 4^6/2 = 2048$$
$$\mathrm{S_M} = \mathrm{V\ S} \quad \text{where } \mathrm{V} = (4!)^6/2 = 95551488$$
$$m_{\mathrm{M}} = \mathrm{P}\ (\mathrm{Q_M})^{a}\ \mathrm{R}^{b}\ (\mathrm{S_M})^{c}$$

$$m_M = m_D\, m$$

$$m_D = T^a V^c$$

Parameter m_M defines the number of reachable states for cubes with marked centres. Factor m_D gives the number of different arrangements of unmarked centre cubies that will provide a solved size n cube. Parameter m_M defines the number of reachable states for cubes with marked centres. It is also the factor by which the number of different states for a standard cube needs to be multiplied by when marked centres apply. My reference[9] provides numerical values for m and m_M for a range of cube sizes n.

The above results for cubes with marked centres have major implications for solving such cubes and some of these are examined in Sec. 8.

6.3 Reachable States for Cubes of All Sizes Simplified

A simplified function[9] for the number of cube states possible for a cube of size n results if that number is expressed in logarithmic form.

Define $m = 10^y$ (or $y = \log_{10} m$).

$y = An^2 + Bn + C$ where A, B and C are constants.

Constants A and B are the same for n even and for n odd but the value of C is different. The constants have the following values. For standard cubes with unmarked centres the following values apply.

$$\begin{aligned} A &= 3.87785955497335 \\ B &= -3.61508538481188 \\ C_{EVEN} &= -1.71610938550614 \\ C_{ODD} &= -4.41947361312694 \end{aligned}$$

Hence, with the logarithmic presentation the number of cube states can be expressed using just four non-integer numbers (A, B and the two C values). Furthermore, the number of cube states form a restricted set of values for a more general continuous parabolic function for which n can have non-integer and negative values. Calculating the value of m from the corresponding value of y is a straight forward process and the above four constants have been validated[9] in respect of known values for m.

For cubes with marked centres the A, B and the two C values are different as shown below.

$$\begin{aligned} A &= 5.87291891862476 \\ B &= -11.59532283941750 \\ C_{EVEN} &= 6.26412806909952 \\ C_{ODD} &= 4.87703443013109 \end{aligned}$$

6.4 Unreachable States for Cubes of All Sizes

The number of unreachable states far exceeds the number of reachable states. There are many references to the number of unreachable states for the size 3 cube but very few for larger size cubes.

The unreachable arrangements for corner and edge cubies are the same for cubes with or without marked centres.

If we consider a corner cubie for cubes of any size then a 1/3 twist clockwise leaving everything else unchanged will represent an unreachable state, and similarly for a 1/3 twist counter-clockwise. Hence only 1/3 of the twist possibilities are reachable.

For the central edge cubie for odd size cubes the behaviour is the same as that for the size 3 cube. Only half the conceivable positions are reachable and only half the conceivable orientations are reachable. Hence only 1/4 of the central edge cubie movement possibilities are reachable.

Edge cubies that comprise 12 complementary pairs (24 cubies total) behave as if the complementary cubies did not look the same. Any given edge cubie can move to any position in the 24-cubie orbit but for any given position there is one reachable and one unreachable orientation for that cubie. The reverse applies for the complementary edge cubie. For a given cubie (1-2) the reachable and unreachable orientations for a given face for a given orbit for a size 8 cube is illustrated below. One of the 24 reachable possibilities for a given edge cubie matches that of the set cube.

Edge cube reachables

			2			1			
			1			2			
1	2							1	2
2	1							2	1
			2			1			
			1			2			

Edge cube unreachables

			1			2			
			2			1			
2	1							2	1
1	2							1	2
			1			2			
			2			1			

The number of unreachable states for a 24-edge-cubie set is the same as the number of reachable states (24! in each case).

As indicated in my reference[9], in the case of the marked centre cubies only half the conceivable arrangements for each set of 24 cubies for any given orbit are reachable. The same parity rules that apply for marked centre cubies also apply for the unmarked centre cubies. A quarter turn of a set-of-four centre cubies cannot be achieved without changing the arrangement elsewhere to meet the parity requirement. Because there are 95551488 (Sec. 6.2) ways of arranging the individual centre cubies so that the resulting arrangement appears exactly the same, parity rules can be met without any observable indication of how the parity compliance is achieved. Hence, for the normal case (24 cubies comprising four of each of six colours) there is no restriction on the achievable states for the centre cubies.

The following table uses the values noted above to represent the *k* component (Sec. 6.1) factors for the size *n* cube. Exponents *a*, *b* and *c* are as defined in Sec. 6.2.

Reduction components for factor k (for standard cube with unmarked centres) and for k_M (for cube with marked centres)	Cube type	
	Standard	Marked centre cubies
Corner cubie factor	3	3
Central edge cubie factor (such cubies exist only for cubes of odd size)	2^{2a}	2^{2a}
Edge cubie factor for all 24–cubie sets combined	2^{b}	2^{b}
Absolute centre cubie factor (such cubies exist only for cubes of odd size)	1	2^{a}
Centre cubie factor for all 24-cubie sets combined	1	2^{c}

For the standard size n cube $\quad k = (3)(2^{2a + b})$

For the marked centres cube $\quad k_M = (3)(2^{3a + b + c})$

Some values for cubes of small size are given below.

Cube size	2	3	4	5	6	7	8
Value of k	3	12	6	24	12	48	24
Value of k_M	3	24	12	192	192	6144	12288

As noted in Sec. 6.1 the number of unreachable states is given by $(k – 1)m$ for standard cubes and by $(k_M - 1)m_M$ for cubes with marked centre cubies.

7. Parity Compliance for Cubes of Size Four and Above

For standard cubes (i.e. cubes with unmarked centre cubies) of size 4 and above, the 24 centre cubies in each orbit have an even distribution of six colours. As indicated in Sec. 6.2, there are many (95551488) ways of rearranging the centre cubies in each orbit that appear exactly the same. In this document parity laws are defined such that they apply independently of any identification colours or markings on cubies. It follows that compliance with parity rules tends to be obscured in standard cubes since position changes of centre cubies of a given colour are not always observable. Compliance with parity rules is readily observable for cubes with marked centres.

7.1 Edge Cubie Final Layer Alignment Issues

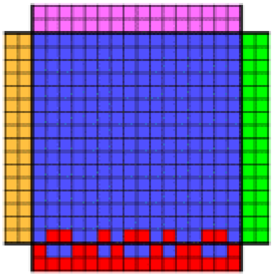
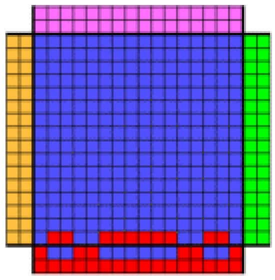
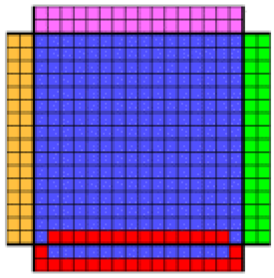
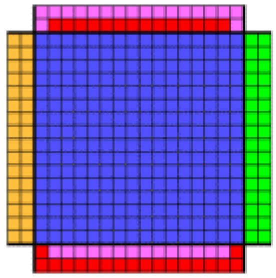
For cubes of size 4 and above it is well known that for the final layer there may be a need for a rearrangement of cubies that cannot be achieved using standard size 3 cube moves. The moves to resolve these issues are well documented but conformity with parity rules can be obscured for standard cubes because the 24 centre cubies in each orbit can be arranged in many ways that look the same but from a parity perspective are different.

While there are many ways of rearranging the cubies to overcome final layer problems that do not arise for the standard size 3 cube, all standard cubes can be solved with attention to two basic problem situations:

- There is a need to flip a complementary pair or a complete set of edge cubies in a final edge set. This condition will be referred to as an OLL (orientation of last layer) requirement.
- There is a need to swap the positions of two edge cubie sets in the final layer. This condition will be referred to as a PLL (permutation of last layer) requirement.

OLL[3] and PLL[4] as used here can be considered to be sub-sets of the usual definitions[10] of these terms. The above needs also arise for cubes with marked centres but additional steps[3] are required to complete the alignment of the final face.

The problems that can arise for a size 16 cube example are illustrated below where departures from a completely solved cube are shown. In practice these problems are likely to be corrected before other alignment actions are completed for the final layer.

1	2	3	4
			

1. Final edge set in this example has three complementary pairs of edge cubies that have a different sense to the other edge cubies in the set.
2. An appropriate sequence of rotations is applied to 1 to correct the problem for one of the complementary pairs (similar sequences are required to perform corrections on the remaining complementary pairs still requiring correction).
3. The sense of all the edge cubie elements in the final set is the same but just this one set needs to be flipped to resolve this problem (this can only occur for cubes of even size). Alternatively, action as indicated here can be avoided by flipping initially the inner complementary pair of edge cubies to match the correct final alignment and then matching all further complementary pairs of edge cubies to the initial pair.
4. The positions of two (and only two) edge cubie sets need to be swapped to resolve this problem (can occur only for cubes of even size). The solution to this problem can also be subdivided into a sequence of moves that applies to just four edge cubies at a time (as for the size 4 cube).

The correction of all of the problems mentioned above normally involves some rearrangement of centre cubies. Problems similar to those illustrated in images 1 and 2 occur in odd size cubes. Rotation sequences required to resolve final layer problems for cubes of any size are similar to those required for the size 4 cube.

7.2 Permutations and Parity for Final Layer Edge Cubie Alignment

Because there are no distinguishing marks on the centre cubies when OLL and PLL corrections are made for standard cubes, it can be difficult to demonstrate how compliance with parity rules is met. Another[2] of my documents, which is primarily concerned with providing instructions and details of the moves that can be used to solve cubes of all sizes, is the source used for defining the small number of moves involved in this analysis. The move notation used here is the same as that used in that reference.

3 OLL usually means manipulating the last layer cubies so that the face has uniform colour, even at the expense of incorrect colours on other sides.

4 PLL usually means moving the last layer cubies to correct positions while preserving their orientation.

Consider the OLL correction for a complementary pair of edge cubies located at the front of the upper layer. The various moves in the above reference are defined in macro[5] terms. The general form for the final edge set alignment macros is:

```
M1c = #cR#cRBBUU#cLUU-#cRUU#cRUUFF#cRFF-#cLBB#cR#cR
```

where `U` etc. means rotate the outer layer of the given face by a quarter turn clockwise.
#cR etc.means rotate just the c^{th} layer from the given face a quarter turn clockwise.
A minus (`-`) ahead of the above means a counter-clockwise quarter turn rotation is involved.
`c` is the column number of the left cubie of the pair to be flipped (c = 1 for the left corner cubie which is not involved in the above move).

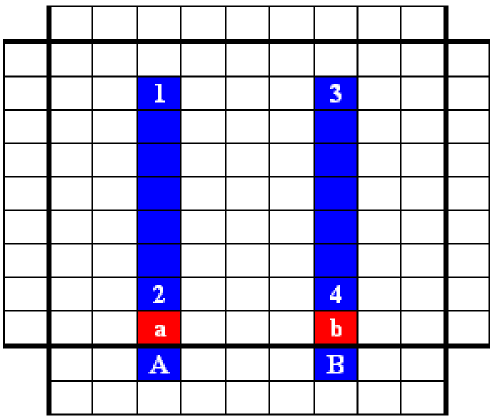
There is no cube-size-dependent item in this macro. The only effect of cube size is the need for more macros of this form. Consider the application of the following OLL macro for a size 9 cube.

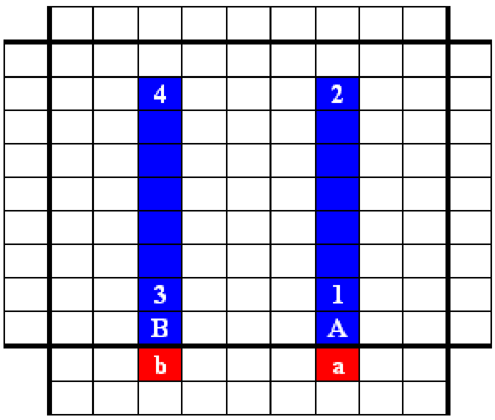
```
M13 = #3R#3RBBUU#3LUU-#3RUU#3RUUFF#3RFF-#3LBB#3R#3R
```

If this macro is applied to a set cube the only observable change is the flipping of a complementary pair of edge cubies. To observe what really happens the macro needs to be applied to a marked cube or a randomised cube or a set cube that has been appropriately disturbed. The effect is clearly visible if applied to a set cube with marked centre cubies[1]. The result is detailed below.

In the following illustrations the only cubies (centre and edge) that move position are shown in colour. Alphanumerical identification shows positional movement.

OLL before correction for size 9 cube	OLL after correction for size 9 cube





For the OLL correction for a size n cube, there are $(n-2)$ centre cubie swaps and overall there are $(n-1)$ swaps when the edge pair is included. For odd size cubes $(n-1)$ is always even (and conforms to the universal parity requirement for odd size cubes). For even size cubes $(n-1)$ is always odd which means in this case a parity reversal always occurs, an allowable parity condition for even size cubes.

5 The term "macro" as used in computer science is a rule or pattern that specifies how a certain input sequence should be mapped into an output sequence. Macros are normally used to map a short string to a longer string (sequence of instructions). Macros simplify things by providing short-cuts for long sets (or frequently used short sets) of rotational instructions to produce a desired change of cube state.

The parity of the above OLL move can also be readily assessed by examining the algorithm used. Since all moves for odd size cubes have even parity, it is relevant to consider only even size cubes. The above OLL move has an even number of outer layer rotations so those rotations have no impact on the move's parity. There is an odd number of inner layer rotations that will always render odd parity for the above OLL move for even size cubes.

For the complete edge set flip (a requirement that can arise only for cubes of even size), there will be (n – 1)(n/2 – 1) swaps. The overall number of swaps will be even if (n/2 – 1) is even (i.e. n/2 is odd). The overall number of swaps will be odd if n /2 is even.

The following macro (permutation) can be used to flip a complete edge set located at the front of the upper layer.

```
M10 = cRcRRRBBUUcL-LUU-cRRUUcR-RUUFFcR-RFF-cLLBBcRcRRR
```

where `c` is equal to half the cube size.
cR etc. means rotate all c layers from the given face a quarter turn clockwise.
A minus (`-`) ahead of the above means a counter-clockwise quarter turn rotation is involved.

For the complete edge set flip, permutation parity will be odd if n/2 is even and even if n/2 is odd. Examination of the M10 macro gives the same result.

For the size 10 cube:

```
M10 = 5R5RRRBBUU5L-LUU-5RRUU5R-RUUFF5R-RFF-5LLBB5R5RRR
```

`M10` has the same outcome as `M12M13M14M15`.

Now consider what is involved in the PLL correction. It can also be considered as being the combined effect of a number of moves involving just four edge cubies as for the size 4 cube. The following macro does a front to back swap for the upper layer.

```
M0c = #cR#cRUU#cR#cRUUdUdUUU#cR#cRdUdUUU
```

where `c` is the column number of the left edge cubie involved in the swap (c = 1 for the left corner cubie which is not involved in the above move).
`d` is equal to half the cube size.

The equivalent macro to perform a left to right swap on the front layer is:

```
M0c = #cU#cUFF#cU#cUFFdFdFFF#cU#cUdFdFFF
```

Consider the application of the following PLL macro for a size 10 cube.

```
M03 = #3U#3UFF#3U#3UFF5F5FFF#3U#3U5F5FFF
```

If this macro is applied to a set cube the only observable change is the swapping of a complementary pair of edge cubies on the left side front face with another complementary pair on the right side of the front face. To observe what really happens the macro needs to be applied to a randomised cube or a marked cube or a set cube that has been appropriately disturbed. The effect is clearly visible if applied to a set cube with marked centre cubies[1]. The result is detailed below.

In the illustrations below the only cubies (centre and edge) that move position are shown in colour. Alphanumerical identification shows positional movement.

For the specific PLL permutation used here, there are $2(n-2)$ centre cubie swaps and overall there are $2(n-1)$ swaps when the edge pairs are included. Hence even parity is always maintained.

The following macro (permutation) can be used to flip a complete edge set located at the front of the upper layer.

```
X = dRdRRRUUdRdRRRUUdUdUUUdRdRRRdUdUUU
```

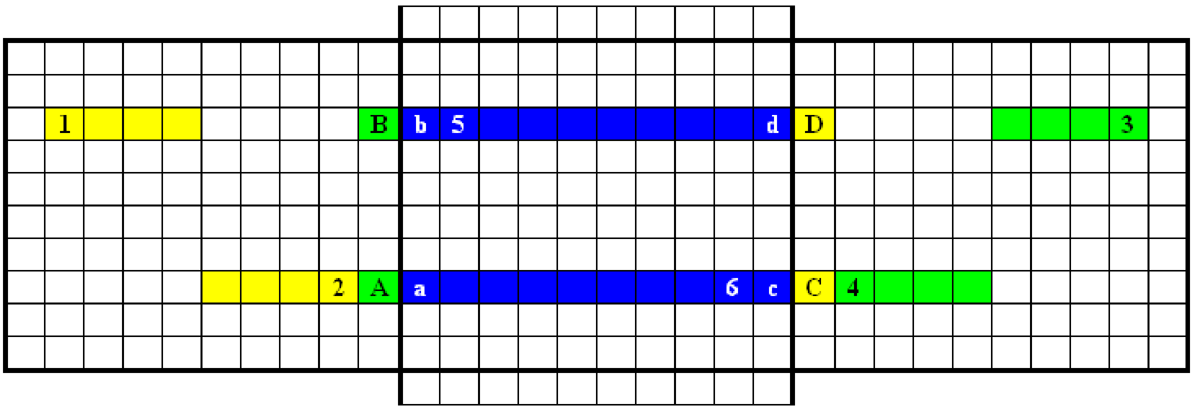
where `d` is equal to half the cube size.

For the size 10 cube:

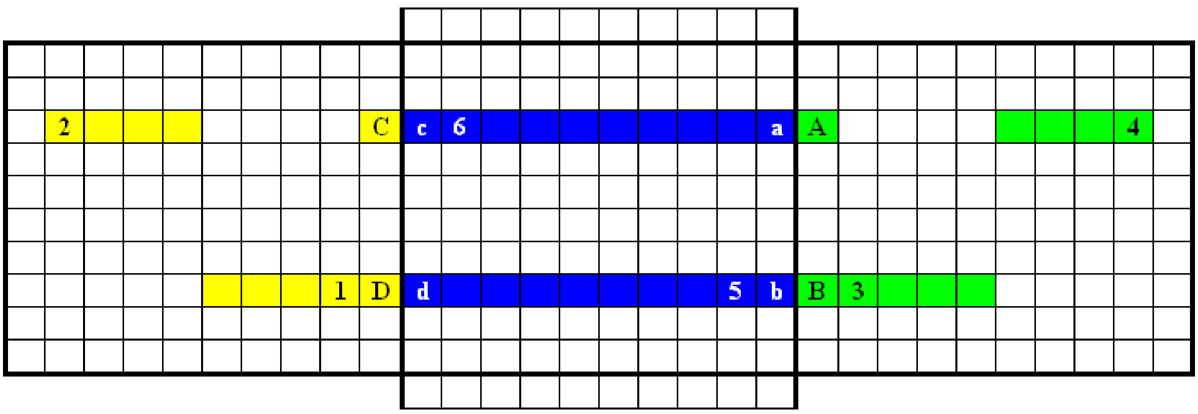
```
X = 5R5RRRUU5R5RRRUU5U5UUU5R5RRR5U5UUU = M02M03M04M05
```

If the complete set of edge cubies is swapped there will be $(n-2)^2$ centre cubie swaps. Adding the $(n-2)$ edge cubie swaps gives a total of $(n-1)(n-2)$ swaps for the above permutation.

PLL before application of M03 correction for a size 10 cube



PLL after application of M03 correction for a size 10 cube



7.3 Further Parity Observations for Standard and Marked Cubes

The OLL and PLL corrections used in the previous section have the following properties:

- If they are applied to a standard cube that has centre cubies aligned, no change to the cube state except for the edge cubies being aligned (flipped and/or moved) is observed after they are applied.

- If they are applied to a standard cube with scrambled centre cubies, or a cube with marked centres, an observable change in the state of the centre cubies occurs after they are applied.

One may pose the question: Does a permutation exist that will result in no observable change to the cube state for standard cubes, except for the edge cubies being aligned, for the general scrambled cubie case? Since there are no unreachable positions for centre cubies in unmarked cubes we may suspect that the answer should be in the affirmative but it is not too difficult to validate it for a specific case.

To simplify matters consider the size 4 cube which has 24 centre cubies comprising four of each of six colours. In effect, larger cubes behave the same for each separate orbit comprising 24 centre cubies for cubes with unmarked centres. Validation can be conveniently performed using a software cube for which cube state can be saved and edited. Scramble the cube and rearrange a pair of edge cubies off-line to conform with the OLL realignment as indicated below. Note that such a rearrangement on-line normally takes place with a rearrangement of centre cubies that may be unobservable.

Change

a	c
b	d

to

d	b
c	a

Define S_{11} as the state after the scramble and S_{21} as the state after the off-line modification. It is known that the cube is solvable from the S_{11} state. Define the overall permutation to solve the cube from state S_{11} to be P_1 and define the resultant solved state as S_{12}. For the size 4 example it was found that the cube is also solvable from state S_{21}. Define S_{22} as the solved state with the same spatial orientation as S_{12}. Define P_2 as the permutation to transform S_{21} to S_{22}.

It follows that, for cubes with unmarked centres, permutation $P_1P_2^{-1}$ or $P_2P_1^{-1}$ where the -1 index signifies the reverse permutation sequence, will perform the OLL alignment without any external observable change to any other cubies. Furthermore, P_1^{-1} applied to S_{12} will not only restore the S_{11} state but the centre cubies will be in exactly the same positions they were in originally. The same applies when P_2^{-1} is applied to S_{22}. To comply with parity rules for cubes of odd size, state S_{21} can only be obtained after rearrangement of the centre cubies, even if the rearrangement is obscured. The behaviour of cubes of even size is more difficult to predict as odd parity can exist. It was observed that the final stage marked centre cubies can be aligned for a size 4 cube with edge cubies misaligned. Hence, to comply with cube rules state S_{12} would not normally be the same as state S_{22} if all 24 centre cubies had distinguishing marks. As indicated in Sec. 6.2, there are 95551488 ways of arranging the centre cubies in a particular orbit for an unmarked cube so that the resulting arrangement appears exactly the same.

A solvable cube results after the following off-line editing actions for a cube with unmarked centres:

- A quarter turn of any set-of-four centre cubies (any mixture of colours allowed) with no change to any other cubies.
- Two centre cubies (having any colour) located anywhere in the same 24 cubie orbit are swapped.
- Complementary edge cubies located anywhere are both flipped.

An unsolvable cube always results after the following off-line editing action for any cube (standard or marked):

- Just one edge cubie is flipped.

The effect of an off-line edge cubie swap and flip was also examined for a marked cube. Except for the size 4 cube the final layer centre cubies cannot be fully aligned if one or more complementary pairs of edge cubies have not been aligned. The results are shown in the following section.

7.4 Marked Cube Solving Note

The results that have been presented in this document have major implications for solving cubes with marked centres. With edge cubies fixed in position, movement of centre cubies is always limited to an even number of swaps. That immutable law can give rise to solving difficulties if some conditions are not met.

1. Due to the interaction between complementary edge cubies and centre cubies it will be impossible to place the final layer centre cubies if the 24 edge cubie sets comprising 12 complementary pairs are not properly aligned for each orbit for cubes of size greater than 4.
2. A missed alignment, for example a need for a 2 to be swapped with a 3 somewhere external to the final face in a particular orbit, will mean the centre cubies in the final face cannot be aligned until the error is corrected. If there is an even number of such erroneous swaps required external to the final layer then the final layer can be fully aligned but to achieve the single final solved state they need to be corrected.

The likelihood that either of these conditions would arise when a marked cube is being aligned increases with cube size. However, the first condition is far less likely to arise than the latter because the color mismatch is far more readily observable than a marking error.

An example of a condition 1 misalignment would be a single complementary pair of edge cubies needing to be swapped. Such a misalignment has a dramatic effect on the last face to be aligned and will normally render it impossible to align the centre cubies on the last face. The centre cubies on the last face can always be aligned for a size 4 cube having a misalignment of edge cubies but realignment of centre cubies will be needed after the edge cubie misalignment is corrected. For that size cube a single swap of a pair of edge cubies is possible since odd parity is possible for cubes of even size. The effect of the edge pair misalignment for cubes of various size is illustrated in the following table. We define here an alignable orbit as one that can return a correct 1234 sequence on the final face and a non-alignable orbit as one that can't. To satisfy the universal even parity rule for odd size cubes the number of non-alignable orbits for cubes of odd size must always be odd.

Cube size (n)	Number of centre cubie orbits (c)	Alignable orbits (w)	Non-alignable orbits (x)
4	1	1	0
5	2	1	1
6	4	2	2
7	6	3	3
8	9	5	4
16	49	25	24
32	225	113	112

The result can be expressed in simple mathematical form.

From Sec. 6.2: $c = ((n - 2)^2 - a)/ 4$

$x = (c - c \bmod 2)/2$

$w = c - x$

Because misalignment of edge cubies has such a dramatic effect on centre cubie alignment it is advisable that centre cubies on all but the first face be aligned after the edge cubies have been aligned.

A condition 2 misalignment can fairly readily arise in cubes of large size because of the small size of the numerals and the fact that there is no change in colour to highlight the misalignment. For example, consider a 2 and 3 need to be swapped somewhere in a particular orbit on a particular face. In that case it will make it impossible to obtain a 1234 alignment of that orbit on the final face until the error has been corrected. In general an odd number of such corrections for a particular orbit will result in the inability to obtain a 1234 alignment whereas an even number will allow alignment. However, corrections are always necessary to achieve the one and only solved state.

8. Concluding Comments

- The set of rules that apply to the standard size 3 Rubik's cube is a special case of a generalised set of rules that are defined herein for cubes of all sizes.
- The standard configuration for centre cubies, where an equal number of six colours is used per 24 centre-cubie-orbit, is a special case of a cube where all such 24 centre cubies have individually identifiable marks.
- Parity rules for standard cubes and those with marked centres are the same but compliance with these rules tends to be obscured for the former.
- Alignment of the final layer is impossible if edge cubies have not been completely aligned before the final layer centre cubies for cubes of size greater than four with centre cubie marking applied.
- The number of states possible for cubes of all sizes can be expressed in terms of four non-integer constants by using a logarithmic presentation.
- The logarithm of the number of possible cube states is a special case of a quadratic (parabolic) function for which Rubik family cube values form a restricted set.

References

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