Rubik's Cube Extended: Derivation of Number of States for Cubes of Any Size and Values for up to Size 25x25x25

by

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Summary

This publication arose as an adjunct to software I had developed that allowed cubes of size 2 x 2 x 2 and above to be solved. General relationships for the number of different states possible for an $N \times N \times N$ cube are derived herein with numeric data calculated for cubes up to size 25 x 25 x 25. The key to presenting the data was to work with the logarithm of the number of states, rather than the number of states themselves, which become so large as to be mind-boggling and not capable of being plotted (presented in graphical form). It is shown that the logarithm of the number of states as a function of cube size lies on a simple parabolic curve. To be rigorous, odd-size cube values lie on a different parabola to that for even-size cube values. However, the difference is constant and is virtually imperceptible when plotted over the cube size range considered (2 x 2 x 2 to 16 x 16 x 16 for graphical plots provided herein).

The original version of this document was written in 1991 but was not published at the time. The document has undergone multiple revisions since then.

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1. Introduction

Although the interest in Rubik's cube peaked in the 1980s, interest remains strong with physical cubes ranging from size 2 x 2 x 2 to 7 x 7 x 7 now being readily available. By far the major interest people have is in solving the cube, but a select minority are also interested in cube mathematics. The only aspect of cube mathematics to be considered herein is the number of different states (permutations, arrangements or combinations) possible in cubes of various sizes. The solved (or set) state is just one of the possible (reachable) states of the cube. Some basic mathematical knowledge is all that is required in reading this article. Be assured that getting the mathematics right has nothing to do with solving the cube.

My interest in calculating the number of different states applicable to cubes of various sizes arose because, back in 1991, I had developed computer software for allowing cubes up to size 15 x 15 x 15 (11 x 11 11 for monochrome monitors) to be manipulated. Subsequently two versions of the software, an earlier C++ version and a later Java version, have been developed. The C++ version accommodates the cube size range 2 x 2 x 2 to 16 x 16 x 16 whereas the Java version has the same lower limit but has a settable upper limit ranging upwards from 16 x 16 x 16 but never exceeding 99 x 99 x 99[1]. Both the C++ and Java versions of the software can be accessed from my website [1]. The Java version also has an option for including marked centres in the 3 x 3 x 3 to 32 x 32 x 32 range typically when numerical marking is used and up to 99 x 99 x 99 when a corner marking extension applies. Apart from the on-line help provided for solving these cubes, my references [2] and [3] give more detailed instructions for standard cubes and for those with marked centres respectively.

Before proceeding, it may be worth expanding upon what is meant by the "number of different states". Basically, if the cube looks different, we have a different state, and if it looks the same we don't have a different state. If you don't perform any twists but just turn the cube around or upside down, that is not classed as a different state. For cubes of size 4 x 4 x 4 and above with unmarked centre elements, there are multiple positions in which individual centre elements can be placed that look identical. While standard cubes with "unmarked" centre elements are considered in detail in this document the change in the number of possible states for marked centre elements is also included (Sec. 7) to complete coverage of the topic. For example, for a solved standard 4 x 4 x 4 cube there are four centre elements on each face that would look the same if the elements were in any of the four positions on each face. In this case, because the different centre cube possibilities look the same, they are regarded as having the same state. It will be shown (Sec. 7) that it is extremely unlikely that a solved cube can be arrived at with all centre elements in the same locations to where they were prior to scrambling of the cube from the set state. If M is the number of different states for a cube of a given size, and all states have an equal probability of occurring after truly random scrambling, then the probability that a scrambler will turn up a set state (or any other defined state) is $1 / M$, which is a miniscule chance, since M is a very large number.

2. Basic Parameters

For convenience the N x N x N cube will be referred to as a cube of size N. Each face will have N x N **face elements** (coloured or otherwise identified "squares"). In every case the cube will have 6 faces and hence the size N cube will have $6N^2$ face elements. The term **cubie** will be used to denote the distinct mini-cubes which are visible from the outer surface of the cube.

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[1] The upper limit is subject to restraints imposed by the monitor's resolution (assuming the full cube is always in view) and the user's visual discrimination. Usually, the number of vertical pixels available sets the limit in respect of resolution.

For a size N cube there will be 8 corner cubies, $12(N-2)$ edge cubies and $6(N-2)^2$ centre cubies. Each corner cubie will comprise elements from three faces, each edge cubie will comprise elements from two faces and the centre cubies will have an element from one face only. The total number of cubies N_C in a size N cube is given in equation 1.

$$N_C = 6(N-2)^2 + 12(N-2) + 8$$
$$= 6(N-1)^2 + 2 \qquad (1)$$

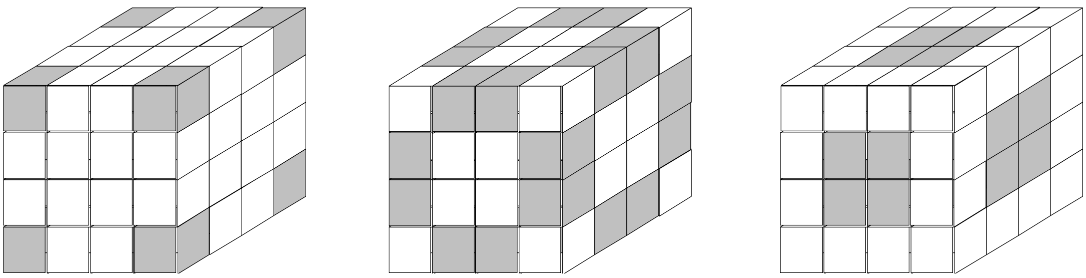
Table 1 provides a summary of the values of the above-defined parameters for cubes in size range 2 to 16 (Fig. 1 shows them for a 4 x 4 x 4 cube). The number of corner cubies is invariant, the number of edge cubies increases linearly as N increases, and the number of centre cubies follows a square law function of N. The number of centre cubies therefore becomes numerically dominant for larger values of N. A unit increase of cube size from N to $N+1$ will increase the total number of cubies by $12N-6$.

Table 1: Basic Cube Parameter Values

Cube size N	Number of Corner Cubies	Number of Edge Cubies	Number of Centre Cubies	Total number of Cubies	Total Face Elements
2	8	0	0	8	24
3	8	12	6	26	54
4	8	24	24	56	96
5	8	36	54	98	150
6	8	48	96	152	216
7	8	60	150	218	294
8	8	72	216	296	384
9	8	84	294	386	486
10	8	96	384	488	600
11	8	108	486	602	726
12	8	120	600	728	864
13	8	132	726	866	1014
14	8	144	864	1016	1176
15	8	156	1014	1178	1350
16	8	168	1176	1352	1536

For my cube solving software, the most significant parameter is the total number of face elements as it defines the size of the storage arrays required to represent a cube of a given size.

Figure 1: Cubies (shaded) for a 4x4x4 cube.



Corner cubies Edge cubies Centre cubies

3. Odd and Even Size Cubes

All cubes of odd size have a central cubie on each face that cannot be moved within the face except to rotate it about its own axis. The relative positions of the six centre cubies never change. Their immobility means one can only shift other cubies around them. Furthermore, the central edge cubie for cubes of odd size can be moved only to another central position. In other words, they behave in exactly the same manner as the edge cubies in the size 3 cube.

For even size cubes of size 4 and above, each edge cubie can move to a complementary position and each centre cubie can move to any one of four positions by rotating the face on which it resides. For odd-size cubes of size 5 and above, non-central edge and centre cubies (except the absolute centre cubie on each face) behave similar to those for even size cubes.

One would therefore expect that the above differences would be reflected when calculating the number of different states possible for odd and even size cubes, and this will be made clear in the following sections.

It is important to identify the numerical factors that, when multiplied together, give the number of different states possible for cubes up to size 4, as these factors are sufficient to define the number of different states possible for cubes of any size.

4. Number of Reachable States for Cubes of Small Size

There are many references that are now available that provide numerical expressions and values for the number of reachable states for cubes of various sizes. However, most supply a result rather than the reasoning required to arrive at the result. Among those that provide their reasoning are Hofstader [4] who, in his 1981 article derives the number of states for the size 3 cube, Wikipedia [5] (including Wikipedia references in that article), which derives the number of states for size 2, 3, 4 and 5 cubes and Jaap [6], who derives the number of states for the size 4 cube. When I wrote the original version of this document in 1991, there was very little written on cube mathematics except for the size 3 cube.

The number of reachable states for a given cube size can be calculated by determining the number of possible arrangements for each type of cubie and then obtaining the product of these numbers.

Irrespective of the size of the cube there will be 8 corner cubies. For a size 3 (or any larger odd size) cube, these can be located in 8**!** (Factorial 8) ways (equal to 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1). Furthermore each corner cubie can have any one of three orientations. At first glance one would expect this would give a further factor of 3^8 However, once seven corner cubies have been placed, the orientation of the last corner cubie is totally defined. Hence the above factor needs to be reduced by a factor of 3 to give a value of 3^7. The possible corner arrangements P_{ODD} is thus given by:

$$P_{ODD} = (8!)\,3^{7} \tag{2}$$

At first glance, one might expect the same value would apply to a size 2 (or any larger even size) cube. However, unlike the size 3 cube, there is nothing to identify the orientation of the cube relative to its internal frame, and the figure derived above for the size 3 cube needs to be reduced by a factor of 24. Refer to Appendix 1 for more detailed information on this assessment.

$$P_{EVEN} = P_{ODD} / 24 = (7!)\,3^{6} \tag{3}$$

The number of possible edge cubie states for the size 3 cube will now be considered. There are 12 such edge cubies and there are two possible orientations for each edge cubie, but account needs to be taken of some restrictions. Of the 12 edge cubies only 10 can be arbitrarily located and hence the number of ways the edge cubies can be located is reduced from 12**!** to (12**!**) / 2 (i.e. reduced by a factor of 2). Furthermore the orientation of 11 edge cubies may be arbitrarily set but then the orientation of the last one is fixed. This results in the orientation factor being reduced from 2^{12} to 2^{11}. The possible edge cubie arrangements Q_{ODD} for the size 3 cube is thus given by:

$$Q_{ODD} = (12!)\, 2^{10} \tag{4}$$

For the size 3 cube the centre cubies cannot move relative to each other. Hence, the number of possible states for the size 3 cube becomes $P_{ODD}Q_{ODD}$.

For the purpose of defining factors for use with cubes of any size, P and Q are defined as follows:

$$P = P_{EVEN} = (7!)\, 3^{6} \tag{5}$$

$$Q = 24\, Q_{ODD} = 24\, (12!)\, 2^{10} \tag{6}$$

Factor P applies to cubes of any size and is the number of states for the size 2 cube. Factor Q applies to cubes of any odd size. The number of possible states for the size 3 cube is equal to PQ. When expanded out:

$$P = 3674160$$

$$PQ = 43{,}252{,}003{,}274{,}489{,}856{,}000$$

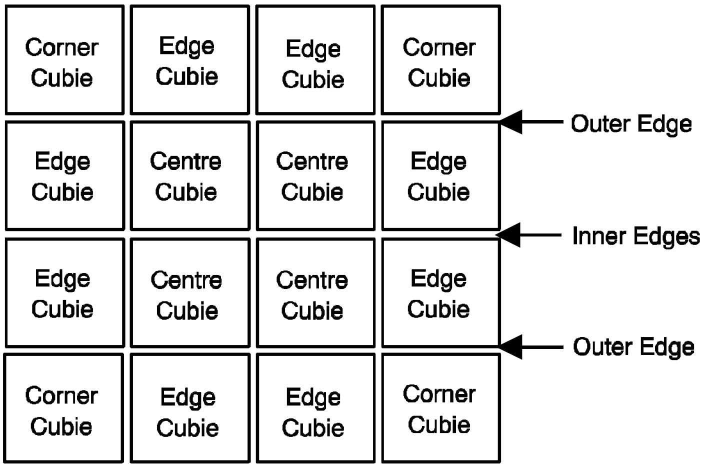
The number of possible states for a size 4 cube will now be considered as two new factors need to be introduced.

For the size 4 cube, the edge cubies are located in pairs and are subject to the obvious restriction that outer and inner edges of the edge cubies can never be interchanged. This fact is illustrated in Figure 2.

The total number of edge cubies for the size 4 cube is 24. For the first pair there will be $^{24}C_2$ ways {(24**!**) / ((22**!**)(2**!**) = (24 x 23) / 2} of selecting a given pair without regard to orientation. Further, each pair can be oriented in two ways. This gives 24 x 23 possibilities for the first pair and so on. In this instance the final two edge pairs can be located in either position and the final edge pair can be arbitrarily oriented. Hence the number of possible edge cubie arrangements R for the size 4 cube is given by:

$$R = 24! \tag{7}$$

Figure 2: Cubie Arrangement for Size 4 Cube Face



Another way of looking at the number of possible edge pair arrangements for the size 4 cube is to consider the overall factor R as being the product of two sub-factors, R_1 and R_2, where R_1 represents the possible arrangements of pairs without regard to orientation or cube position and R_2 represents the possible arrangements of a given set of edge pairs. R_1 thus takes account of the pairing possibilities and R_2 takes account of their arrangement. If, in the derivation of R in the previous paragraph, the orientation of edge cubie pairs were to be ignored, $(24!)/2^{12}$ possibilities would result. If the order of the 12 edge pairs were also ignored, the possibilities R_1 would become equal to $(24!)/\{2^{12}(12!)\}$. This is in effect the pairing possibilities of the 24 edge cubies without regard to position on cube or orientation. The number of possible arrangements R_2 of a given set of 12 edge pairs is equal to $2^{12}(12!)$. R_2 is similar to the factor Q_{ODD} for the size 3 cube but without a location restriction on the final two pairs and an orientation restriction on the final pair as mentioned above for the size 3 cube. As expected, $R_1 R_2 = R$.

The possible states of the centre cubies for a size 4 cube will now be considered. In this case there will be four centre cubies on each face, and a total of 24 for the cube. Unlike the corner and edge cubies, the centre cubies of a given face colour (or other identification) are not unique. Exchange of a centre cubie with any of the remaining three centre cubies of the same face identification will result in exactly the same observed state for the cube. If we were to temporarily place individual markers on each of the centre cubies (e.g. by use of the numbers 1 to 4), we would expect the number of possible arrangements to become 24!. However, for any specific arrangement of edge cubies, only half those states are reachable (a full explanation of this is provided in Appendix 2). Hence the number of possible arrangements for marked centre cubies becomes (24!)/2. If the special markings are now removed, we need to reduce the above number of possibilities by a factor of 4! for each set of four identical centre cubies, except the 4! factor for the last set of four which is halved to account for unreachable states similar to that which applied for marked centre cubies (also refer to Appendix 2 for explanation). Hence the net number S of possible arrangements for the centre cubies of a size 4 cube becomes:

$$S = (24!)/(4!)^6 \qquad (8)$$

The number of possible states for the size 4 cube is equal to PRS. When expanded out:

PRS = 7,401,196,841,564,901,869,874,093,974,498,574,336,000,000,000

5. Number of Reachable States for the Extended Cube

Before deriving the number of reachable states (also referred to as permutations[2] or arrangements) for cubes with more than four layers, it is worth summarising the factors derived for the size 3 and 4 cubes as the number of states for larger size cubes will be functions of these factors.

Table 2: Basic Parameter Values

Corner cubie possibilities for even size cubes	*P*	$(7!)\ 3^{6}$	$3.67416000000000\text{x } 10^{6}$
Central edge cubie possibilities for odd size cubes, multiplied by 24	*Q*	$24\ (12!)\ 2^{10}$	$1.17719433216000\text{x } 10^{13}$
Edge cubie possibilities for each dual set (12 pairs)	*R*	$24!$	$6.20448401733239\text{x } 10^{23}$
Centre cubie possibilities for each quadruple set (6 groups of 4)	*S*	$(24!) / (4!)^{6}$	$3.24667053711000\text{x } 10^{15}$

In general, for cubes of size *N* greater than 4, the edge cubies will form pairs for which factor *R* applies, and for *N* odd, factor *Q* will apply for the set of central edge cubies. The edge cubie arrangement for *N* = 6 and *N* = 7 is illustrated in Figure 3 where 0 signifies a single edge cubie and 1, 2 signify double edge cubie sets.

Figure 3: Sample Single and Double Edge Cubie Arrangements

N = 6

	2	1	1	2	

N = 7

	2	1	0	1	2	

The centre cubies will form sets of 24 (also referred to as orbits) for which factor *S* provides the number of reachable arrangements for each set. For *N* odd, there will always be a single centre-most cubie on each face. Such cubies, like those for the size 3 cube, cannot move relative to the centre-most cubie on any other face. In mathematical terms, the number of centre cubies for *N* even must be exactly divisible by 4, and for *N* odd, the number of centre cubies per face minus one must be exactly divisible by 4. The arrangements of sets-of-4 centre cubies for size 6 and 7 cubes are illustrated in Figure 4 where 0 signifies the centre-most cubie for *N* odd and 1, 2, 3 etc. signify the various sets-of-4 cubies.

Figure 4: Sample Centre Cubie Arrangements on One Face

N = 6

	2	3	4	2	
	4	1	1	3	
	3	1	1	4	
	2	4	3	2	

N = 7

	3	4	5	6	3	
	6	2	1	2	4	
	5	1	0	1	5	
	4	2	1	2	6	
	3	6	5	4	3	

The number of reachable states *M* for a size *N* cube can now be defined in terms of the factors *P*, *Q*, *R* and *S*.

$$M = PQ^{a}R^{b}S^{c} \tag{9}$$

2 The word "permutation" is used with slightly different meaning in various texts. It appears to be used to mean "state", "change of state" or the "act of changing a state".

where a = $N \bmod 2$ (i.e. 0 if N is even or 1 if N is odd)[3]

b = $0.5\,(N - 2 - a)$

c = $0.25\,((N - 2)^2 - a)$

Parameters P, Q, R and S are integer constants. For the Rubik's cube family a, b and c are integer variables (functions of cube size N). It is convenient to retain Q and S as separate entities as their values are different for cubes with marked centres (considered later).

The above equation for M agrees exactly with the first formula given by Hardwick [7] provided that only the integer portion of the indices in that reference is used for odd size cubes. Parameter M may also be expressed as: $M = 10^Y$ where $Y = log_{10}\,M$.

Numeric values of a, b, c, PQ^a, R^b, S^c, M and Y are provided in Table 3 as a function of cube size N in the range 2 to 25. However, the above formulae are valid for any value of N greater than 1. Number crunching and plotting were done with Excel 2003[4] in companion document *cubecalc.xls*. Calculations were done using a quasi-logarithm approach. If raw numbers are used, numeric overrun will occur for N greater than 9. Separate calculations on number base (where 1 < number base < 10) and on exponent avoids overruns. The calculations were performed to 15 significant figures, which is Excel's limit. Values of PQ^a, R^b, S^c and M given in the table have been truncated to 4 decimal places. The value of Y is also tabulated. The table clearly shows that the Quadruple Centre Cubie factor S^c becomes the dominant contributor for cubes of size 5 and above. Values of M and Y to the full 15 significant figures are provided in Appendix 3.

The calculations for cubes in the size range 2 to 16 are presented graphically in Fig. 5 in the form of the logarithm Y of the reachable states M as a function of the cube size N. The value of M becomes very high as N increases. The rate of increase, on a logarithmic scale, increases linearly with N.

Table 3: Reachable States for N = 2 to N = 25

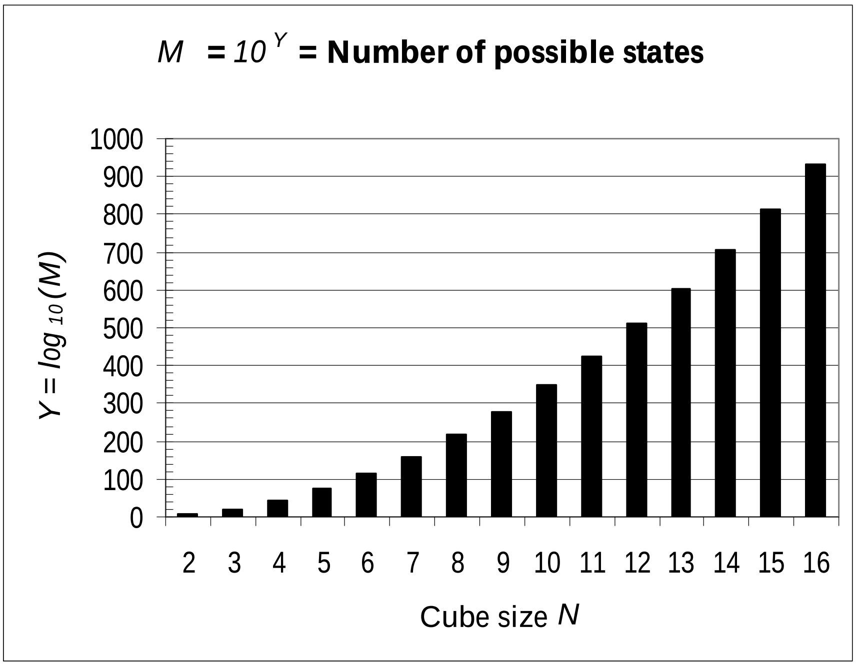
N	a	b	c	PQ^a	R^b	S^c	M	Y
2	0	0	0	3.6741×10^{6}	1.0000	1.0000	3.6741×10^{6}	6.565
3	1	0	0	4.3252×10^{19}	1.0000	1.0000	4.3252×10^{19}	19.636
4	0	1	1	3.6741×10^{6}	6.2044×10^{23}	3.2466×10^{15}	7.4011×10^{45}	45.869
5	1	1	2	4.3252×10^{19}	6.2044×10^{23}	1.0540×10^{31}	2.8287×10^{74}	74.452
6	0	2	4	3.6741×10^{6}	3.8495×10^{47}	1.1110×10^{62}	1.5715×10^{116}	116.196
7	1	2	6	4.3252×10^{19}	3.8495×10^{47}	1.1711×10^{93}	1.9500×10^{160}	160.290
8	0	3	9	3.6741×10^{6}	2.3884×10^{71}	4.0081×10^{139}	3.5173×10^{217}	217.546
9	1	3	12	4.3252×10^{19}	2.3884×10^{71}	1.3716×10^{186}	1.4170×10^{277}	277.151
10	0	4	16	3.6741×10^{6}	1.4819×10^{95}	1.5240×10^{248}	8.2983×10^{349}	349.919
11	1	4	20	4.3252×10^{19}	1.4819×10^{95}	1.6934×10^{310}	1.0854×10^{425}	425.036
12	0	5	25	3.6741×10^{6}	9.1945×10^{118}	6.1087×10^{387}	2.0636×10^{513}	513.315
13	1	5	30	4.3252×10^{19}	9.1945×10^{118}	2.2036×10^{465}	8.7635×10^{603}	603.943

3 If h and n are positive integers, h modulo n (abbreviated to "h mod n") is the remainder that results if h is divided by n. Various programming languages (e.g. Java) express h mod n in terms of the Modulus operator "%" (h % n).

4 Calculations to higher precision than that shown in this document are available in the companion document *cubecalc.xls*. That document is also the source of the graphical plots reproduced in this document.

14	0	6	36	3.6741×10^{6}	5.7047×10^{142}	2.5809×10^{558}	5.4096×10^{707}	707.733
15	1	6	42	4.3252×10^{19}	5.7047×10^{142}	3.0227×10^{651}	7.4583×10^{813}	813.873
16	0	7	49	3.6741×10^{6}	3.5395×10^{166}	1.1494×10^{760}	1.4948×10^{933}	933.175
17	1	7	56	4.3252×10^{19}	3.5395×10^{166}	4.3706×10^{868}	6.6909×10^{1054}	1054.825
18	0	8	64	3.6741×10^{6}	2.1961×10^{190}	5.3957×10^{992}	4.3536×10^{1189}	1189.639
19	1	8	72	4.3252×10^{19}	2.1961×10^{190}	6.6612×10^{1116}	6.3271×10^{1326}	1326.801
20	0	9	81	3.6741×10^{6}	1.3625×10^{214}	2.6699×10^{1256}	1.3366×10^{1477}	1477.126
21	1	9	90	4.3252×10^{19}	1.3625×10^{214}	1.0701×10^{1396}	6.3066×10^{1629}	1629.800
22	0	10	100	3.6741×10^{6}	8.4539×10^{237}	1.3926×10^{1551}	4.3255×10^{1795}	1795.636
23	1	10	110	4.3252×10^{19}	8.4539×10^{237}	1.8122×10^{1706}	6.6262×10^{1963}	1963.821
24	0	11	121	3.6741×10^{6}	5.2452×10^{261}	7.6564×10^{1876}	1.4755×10^{2145}	2145.169
25	1	11	132	4.3252×10^{19}	5.2452×10^{261}	3.2348×10^{2047}	7.3386×10^{2328}	2328.866

Figure 5: Reachable States for Size *N* Cube



6. Logarithm of Number of Reachable States as Function of Cube Size

In the previous section the number of possible states *M* was first calculated and then the logarithm of *M* was used to produce something that could be plotted. In this section we will derive the generalised relationship between $log_{10}\, M$ and cube size *N.* The simplicity of this approach will soon become apparent. The logarithm of the number of reachable states *Y* may be represented as a continuous function of *N* subject to the Rubik's cube family restriction that *N* must be an integer greater than 1. Because the continuous functions we will be using are valid for any value of *N* (including non-integer, and both positive and negative values), we will represent the previously used upper case representation for *N*, *M*, and *Y* with a lower case representation. Assume *a* retains 0 or 1 value. From equation 9:

$$y = log_{10}(m)$$
$$= A\,n^{2} + \mathrm{B}\,n + C \qquad (10)$$

where A	=	$0.25\ log_{10}\ S$	(11)
B	=	$0.5\ log_{10}\ R - log_{10}\ S$	(12)
C	=	$log_{10}\ P + a\ (log_{10}\ Q) - (1 + a/2)\ log_{10}\ R + (1 - a/4)\ log_{10}\ S$	(13)
C_{EVEN}	=	$log_{10}\ P - log_{10}\ R + log_{10}\ S$	(14)
C_{ODD}	=	$log_{10}\ P + log_{10}\ Q - 1.5\ log_{10}\ R + 0.75\ log_{10}\ S$	(15)
C_{DIFF}	=	$C_{EVEN} - C_{ODD} = -log_{10}\ Q + 0.5\ log_{10}\ R + 0.25\ log_{10}\ S$	(16)

Coefficients A and B are constant, and C is constant except that its value for odd size cubes is different to its value for even size cubes. Values for these constants were calculated using Excel 2003 and the resulting values are given below.

A	=	3.87785955497335
B	=	–3.61508538481188
C_{EVEN}	=	–1.71610938550614
C_{ODD}	=	–4.41947361312694
C_{DIFF}	=	2.70336422762081

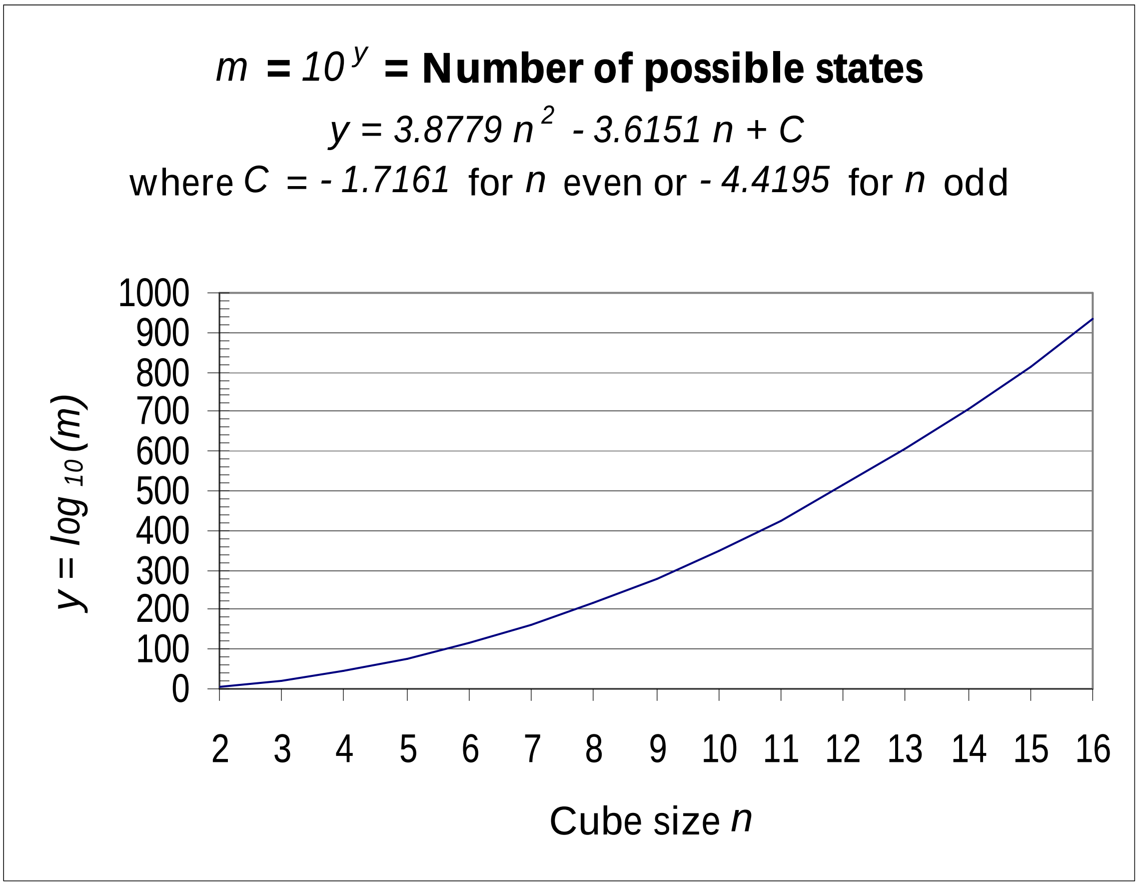
The above analysis indicates the logarithm of the number of states *y* is a quadratic function of cube size *n*. When plotted as a continuous curve (variable *n* not restricted to integer values), such a function produces a parabola. Strictly speaking, two parabolas are involved, one on which even-size cube values lie and another on which odd-size cube values lie. The parabolas are separated by the constant value C_{DIFF} (equal to 2.70336422762081) with the "even" parabola sitting above the "odd" parabola. The parabolas are of exactly the same shape. The difference between the curves translates as a factor equal to 10 raised to power C_{DIFF} (also equal to $R^{0.5} S^{0.25} / Q$), which yields a value of 505.08471690483. *y* is plotted as a continuous function of *n* in Figure 6. The C_{EVEN} value was used for plotting the "n even" points and the C_{ODD} value for the "n odd" points in the Figure 6 plot. These figures match those in Table 3. However, the difference between the "even" parabola and the "odd" parabola is imperceptible when the parabolas are separately plotted for the cube size range considered.

The result that the "even" parabola is positioned slightly above the "odd" parabola is to be expected as a consequence of the immobility of the centre cubie on each face and the limited movement of the central edge cubie for cubes of odd size.

An approximate single parabola for both "even" and "odd" size cubes is one that lies midway between the two ($C = 0.5(C_{EVEN} + C_{ODD}) = -3.0678$). An approximate function for any *n* is:

$$y = 3.8779\ n^2 - 3.6151\ n - 3.0678 \qquad (17)$$

Figure 6: Logarithm of Reachable States as Continuous Function



The slope $\frac{dy}{dn}$ of the y-*n* curve increases as a linear function of *n* as given in equation 18.

$$\frac{dy}{dn} = 7.7557n - 3.6150 \tag{18}$$

Physically, this means that the change in *y* for a unit change in *n* increases as a linear function of *n*. The vertex (slope = 0) of both the even and the odd parabolas is located at *n* = 0.46611865818806, for which *y* = -2.55863875990804 for *n* even or *y* = -5.26200298752884 for *n* odd. The separation between the "even" and "odd" parabolas is observable when the parabolas are plotted in the vicinity of their vertices as shown in Figure 7. The vertices are, of course, outside the range of interest for the cube. The only valid points representing the number of cube states on the parabolas of Figure 7 are the value for *n* = 2 on the "even" parabola and the value for *n* = 3 on the "odd" parabola.

Since $log_{10}(m)$ is easier to work with than *m* itself, the preferred method for calculating *m* would be to derive it from $log_{10}(m)$. For the large numbers involved, equation 19 (below) could be used with Excel for example.

Using

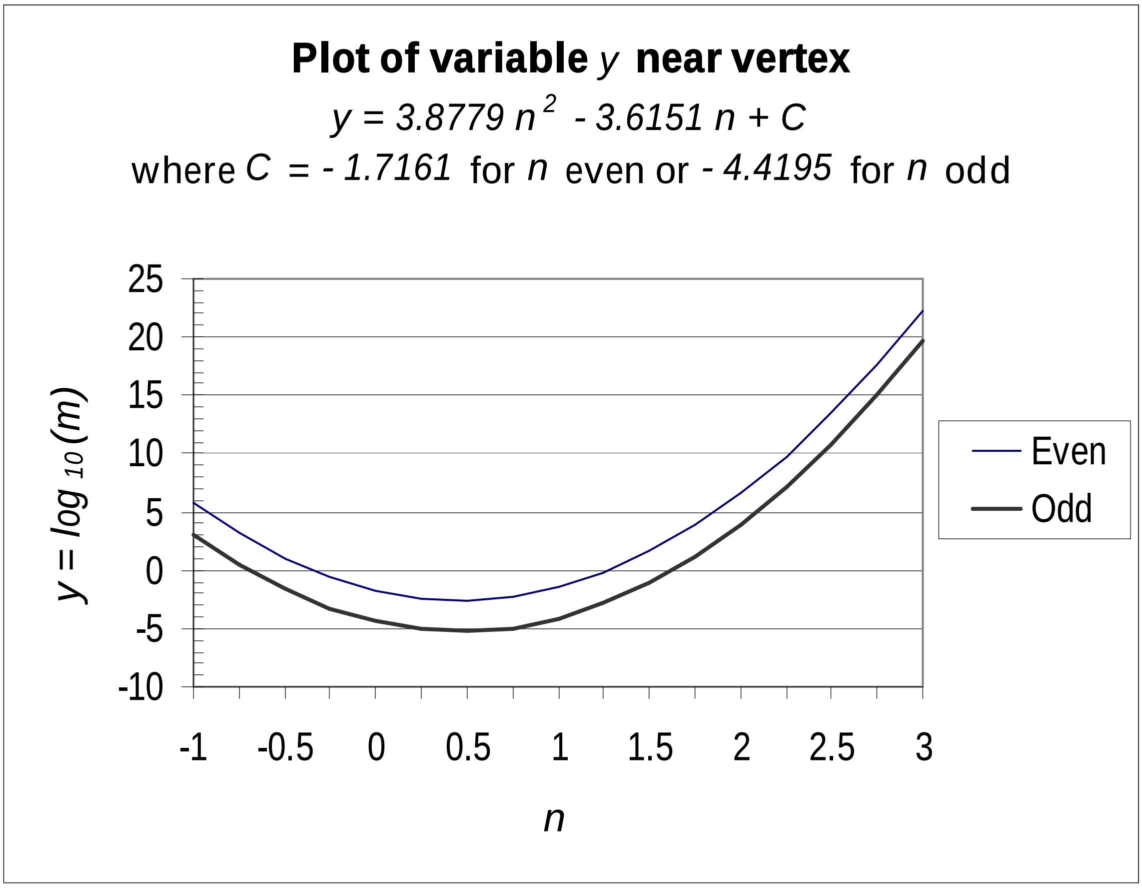
$$y = log_{10}(m)$$

$$m = antilog_{10}.(y - INT(y)) \times 10^{INT(y)}$$

$$= 10^{y - INT(y)} \times 10^{INT(y)} \tag{19}$$

where *INT(y)* is the integer portion of *y*

Figure 7: Plot of two parabolas in vicinity of vertex



Various people refer to a size 1 cube. That cube has just one reachable state with the only movement possible being to change its orientation relative to its surroundings (refer to Appendix 1 for further comments). If it were a valid member of the Rubik cube family it would return $m = 1$ and be on the odd parabola at y = 0. For $n = 1$:

$$m = 6.97108787026676 \times 10^{-5} \text{ or } y = -4.15669944296547$$

and hence it is not a valid member of the Rubik cube family. Fig. 7 clearly shows it is not a valid member of the Rubik cube family.

The use of base 10 for the logarithms in this section is convenient but not necessary. Any other base could be used. The resulting curve shapes would be the same but the vertical scaling would be different. For example, if Natural (Naperian) logarithms to base *e* were used and the graphs plotted against *z* on the vertical axis:

$$m = 10^{y} = e^{z}$$

$$z = y / \log_{10} e$$

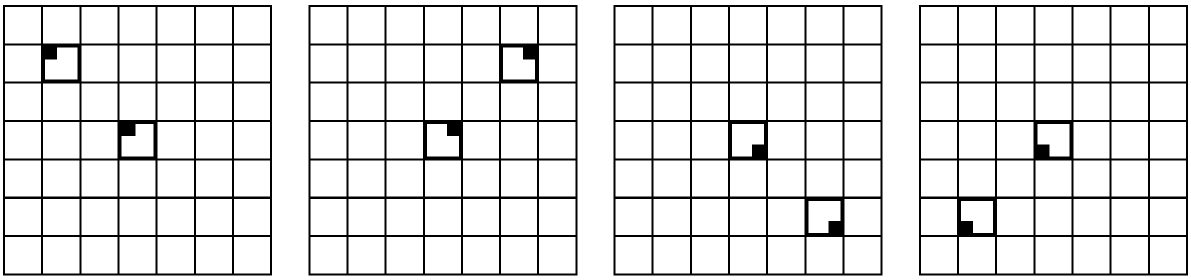
It has been shown in this section that, with the logarithmic presentation, the number of cube states can be expressed using just four non-integer numbers (A, B and the two C values). Furthermore, the number of cube states form a restricted set of values for a more general continuous parabolic function for which *n* can have non-integer and negative values.

7. Number of Reachable States for Cubes with Marked Centre Elements

Centre cube elements are different from corner or edge cube elements in that, unless they have indicative markings, there are multiple possibilities for their final orientation and/or locations. A marked centre cube element has an image or other identification that dictates the correct orientation and position that element must have for a solved or unscrambled cube. Such markings are sometimes applied to hardware cubes of small size but are in very rare use for software cubes, particularly those of large size. There may be some interest in the number of different arrangements of centre cube elements that will produce a solved unmarked size *N* cube.

The absolute centre cubie on any face in an odd-size cube has a fixed facial position but has four possible orientations. The members of each set-of-4 centre cubies (Fig. 4), for cubes of size 4 and above, can occupy any one of four possible locations. In this latter case cubie orientation depends on position and cannot be changed independent of position. The orientation possibilities for the absolute centre cubie and the location possibilities for one of the set-of-four cubies are illustrated in Fig. 8 for a size 7 cube.

Figure 8: Orientation and Location Possibilities for Centre Cubies for Size 7 Cube



When an odd size cube is unscrambled, apart from the orientations of the central cubies, there will always be an even number of quarter turns of these centre cubies required to complete the cube alignment. With the six cube faces, the number of possible configurations *T* of the absolute centre cubies would be 4^6 except that, due to the above condition, we need to halve that result [5][6]. Therefore, *T* is given by equation 20.

$$T = 4^6/2 = 2^{11} = 2048 \qquad (20)$$

Parameter *T* can be included in a new *Q* parameter (Sec. 5) which we will define as Q_M.

$$Q_M = TQ \qquad (21)$$

We now consider the impact of having marked cubies in the sets-of-four centre cubies applicable to cubes of size 4 and above. The number of different centre cubie states for a cube with markings would be 24**!** if no restrictions applied. However, a factor of 2 reduction, (see Sec. 4 comment) as for *T* for the absolute centre cubie for odd size cubes, needs to be applied. (Imagine the centre cubies temporarily stuck together to visualize the correspondence to the absolute centre cubie for odd size cubes.) From Sec. 4, the number of reachable arrangements for unmarked centre cubies is $(24!)/(4!)^6$. The latter value was given the designation *S* in Sec. 4. If we denote S_M for the marked version parameter:

$$S_M = V\,S \qquad (22)$$

where $V = (4!)^6 / 2 = 95551488$

Let the number of reachable states for a size N cube with distinguishable centre cubies be denoted by M_M and let the factor to be applied to M to give M_M be M_D.

$$M_M = P(Q_M)^a R^b (S_M)^c \quad (23)$$

$$M_M = M_D M \quad (24)$$

$$M_D = T^a V^c \quad (25)$$

where *a, b* and *c* are as defined in Sec. 5.

When parameter values are substituted in equation 23, the result is in agreement with the second formula provided by Hardwick [7] provided that only the integer portion of the indices in that reference is used for odd size cubes.

The number of reachable states (M_M) for cubes up to size 25 with distinguishable centre cubies has been calculated together with graphical plots, as for cubes with unmarked centres, in companion document *cubecalc_mark.xls*. Values of M_M and M_D for the size 3, 4, 8 and 16 cubes are given below.

N	M_M	M_D
3	$8.85801027061552 \times 10^{22}$	$2.04800000000000 \times 10^{3}$
4	$7.07195371192427 \times 10^{53}$	$9.55514880000000 \times 10^{7}$
8	$2.33537241311899 \times 10^{289}$	$6.63952623752378 \times 10^{71}$
16	$1.60770294148818 \times 10^{1324}$	$1.07556156818020 \times 10^{391}$

Factor M_D gives the number of different arrangements of centre cubies that will provide a solved unmarked size N cube. If only cubie location differences are considered, the number V^c applies. If one starts with a set (solved) unmarked cube, scrambles it and then solves it, then the probability that the centre cubies in the solved cube will have the same position and orientation as they had in the original set state is $1 / M_D$, a miniscule value. For the simple size 3 cube the probability is 1 / 2048 meaning that one would expect all centre cubie orientations in the solved unmarked cube would match all orientations of the cube prior to scrambling, on average, only once every 2048 times that the cube is solved.

8. Cube Spatial Orientation Relative to External Environment

All the above considerations of the number of reachable cube states, for both unmarked and marked centre cubies, have assumed that changing the spatial orientation of the whole cube (by turning it upside down for example) does not change the cube state. If a different spatial configuration were to represent a different state, a further factor of 24 would need to be applied for the number of reachable states for cubes of any size. Refer to Appendix 1 for further comments.

9. Concluding Remarks

Even if one could flip through the various state possibilities at the rate of one million per second, it would take about a million years to exhaust all the possibilities for the humble 3-

layer cube! One should not be daunted by such large figures as most are fully aware that the 3-layer cube can be unscrambled in the order of seconds by a cubist familiar with the path required for solving the cube.

Expressions for the number of reachable states for standard cubes of any size have been derived herein and numerical values for cubes in the size range 2 x 2 x 2 to 25 x 25 x 25 have been provided.

The results for standard cubes in the size range 2 x 2 x 2 to 16 x 16 x 16 have been plotted as the logarithm of the number of reachable states as a function of cube size. That provided the only practical way of plotting numbers that vary between such huge limits.

For cubes of size 3 x 3 x 3 and above, the effect on the number of reachable states of having centre elements with distinguishable markings, was also derived. It was shown that there was negligible chance that the positions and orientations of all centre elements in a solved cube with unmarked centre elements would align exactly with those applicable to the cube prior to scrambling.

A great simplification resulted by deriving the relationship between the logarithm of the number of reachable states as a continuous function of cube size. A simple quadratic function resulted with coefficients universally constant except for a small but constant difference between the functions for even-size cubes and odd-size cubes. In effect, the number of reachable states for members of the cube family form a special sub-set of the more general continuous quadratic function.

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11. Scherphuis, J., *Jaap's Puzzle Page*, http://www.jaapsch.net/puzzles/cube3.htm, as accessed on 12 February 2017.
12. Fraser, K., *Rules for Rubik's Family Cubes of All Sizes*, http://www.kenblackbox.com/cube/maths/cuberules.pdf.

Companion Documents

- Fraser, K., Calculations for Extended Rubik's Cube, http://www.kenblackbox.com/cube/maths/cubecalc.xls.
- Fraser, K., Calculations for Extended Rubik's Cube with Marked Centres, http://www.kenblackbox.com/cube/maths/cubecalc_mark.xls.

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Appendix 1

Cube Spatial Orientation

Cube spatial orientation will be examined in relation to two assessments:

1. The alignment of the whole cube relative to the environment external to the cube object.
2. The alignment of corner cubies relative to a reference frame within the cube object.

Cube orientation relative to the external environment is expressed in terms of D (down), U (up), B (back), F (front), L (left) and R (right) faces. There are three axes of rotation for the cube. One can be drawn through the centres of the D and U faces (DU axis). The others are the BF and LR axes. Algorithms that can be defined to assist people to unscramble cubes are always expressed in terms of rotations of layers about these axes. For hardware cubes this set of axes can be considered to remain essentially in fixed orientation relative to the external environment. For software cubes their position on the screen does not change.

A similar set of axes can be defined for cube object itself. These are usually referred to in terms of face colours. Commonly used colours are white (W), blue (B), orange (O), red (R), yellow (Y) and green (G). One axis can be drawn through the centres of the W and B faces (the WB axis). The others are the OR and YG axes.

First consider the number of ways the cube can be presented relative to the external environment. For example, consider first selecting the colour for the bottom face. There are six faces and therefore six possibilities for this selection. For example assume we select the W face. That defines the opposite (up) face B as well. That leaves four remaining faces. Assume we select the R face as the front face. That defines the opposite (back) face O as well. The left face Y and right face G are also set when the front face is chosen. Hence there are 24 ways the cube can be presented relative to the external environment.

For odd size cubes the absolute centre cubies can be considered to reside on a fixed frame within the cube object. There can be no relative positional movement between the absolute centre cubies and they can be considered to form a fixed reference frame within the cube object. Rotation of these cubies about their own axes is possible but that rotation is relevant only when considering cubes with marked centre cubies. All cubie movements in odd size cubes can be considered as rotations relative to the fixed reference frame. For even size cubes no such reference frame can be observed from the external surfaces of the cube. Similar to above there are six ways of making the first colour selection and four ways of making the remaining colour selection. Hence any one of 24 possibilities will be acceptable for even size cubes. This also means that when aligning corner cubies for even size cubes there will be a factor of 24 less possibilities than those that apply for odd size cubes (repositioning the whole cube does not produce a changed state).

Appendix 2

Factor of Two Reduction in Number of Reachable States for Centre Cubies

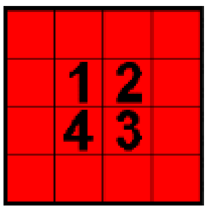
For a size 4 cube there will be one set of 24 centre cubies (comprising a quadruple set on each face). Cubes of size greater than 4 will have a higher number of 24-cubie sets. Movement between the sets of 24 cubies is not possible. Permutation possibilities for each 24-cubie set are identical, so consideration of the number of reachable states for a cube of any size greater than 4 can be extrapolated from what applies for the size 4 cube.

There are many references to the number of unreachable states for the basic Rubik's cube (size 3 cube). Such states can be reached only by removal and replacement of cubies or their stickers in hardware cubes, or by off-line rearrangement of their state for software cubes, such that the rearranged state would be unreachable for a normal cube. For example, website references [5], [6], [8], [9] and [10] examine unreachable states for the standard size 3 cube with unmarked centre cubies, while website references [5] and [11] examine additional unreachable states for the size 3 cube with marked centres. My reference [12] derives the number of unreachable states as a function of cube size for both unmarked and marked cubes.

Most websites imply that the number of reachable states for centre cubies with identifying markings for a size 4 cube is 24**!**. However, for any specific arrangement of edge cubies, half those states are unreachable. For the size 3 cube with identifying marks to indicate the correct orientation of each centre cubie for the solved cube, it was indicated in Sec 7 that if all the cube except for the orientation of the centre cubies has been unscrambled, there will always be an even number of centre cubies requiring a quarter turn. For a size 3 cube with marked centres the number of reachable states of the centre cubies would be 4^6 except that, due to the above condition, we need to halve that result. In effect, if we consider a particular centre cubie then if the state of everything else in the cube is to remain unchanged, then only two states are possible – the current centre cubie orientation and a half turn relative to that orientation. While this condition always applies, it is usually focussed on the last centre cubie to be placed. The user will observe that only two possibilities exist – the correct orientation and a half turn relative to that.

The size 4 cube subject to only outer layer rotations will behave exactly like a size 3 cube and hence be subject to the "even number of centre cubies requiring a quarter turn" condition and a similar reduction in the reachable arrangements as for the size 3 cube will apply.

For a size 4 cube with identifying markings on all centre cubies, there would be 24**!** permutation possibilities if all states were reachable. Consider a cube with 1-2-3-4 centre cubie markings [3] as illustrated below.



If the arrangement shown is that for the set state (i.e. such an arrangement is reachable) then the only other reachable states, assuming everything except this face is aligned to its final state, will be those that can be reached from an even number of swaps as indicated in the following table.

Sequence	Swaps	Parity	Reachable
1234	0	even	Yes
1243	1	odd	No
1324	1	odd	No
1342	2	even	Yes
1423	2	even	Yes
1432	1	odd	No
2341	3	odd	No
2314	2	even	Yes
2413	3	odd	No
2431	2	even	Yes
2134	1	odd	No
2143	2	even	Yes
3124	2	even	Yes
3142	3	odd	No
3214	1	odd	No
3241	2	even	Yes
3412	2	even	Yes
3421	3	odd	No
4123	3	odd	No
4132	2	even	Yes
4213	2	even	Yes
4231	1	odd	No
4312	3	odd	No
4321	2	even	Yes

Hence for the last face to be set, only 12 of the 24 hypothetical possibilities are reachable. Hence for a size 4 cube with marked centre cubies there will be $24!/2$ reachable states.

If the special markings are now removed, we need to reduce the above number of possibilities by a factor of $4!$ for five of the six sets of four identical centre cubies and the last set by $4!/2$ for the reasons given above. Hence the reduction factor is $(4!)^6/2$.

Taking into account unreachable states, the reachable unmarked centre cubie possibilities for the size 4 cube becomes $24!/(4!)^6$. Hence the factor-of-two reductions cancel for the size 4 cube with unmarked centres so the correct result still applies if these reduction factors are ignored. However, ignoring the factor-of-two reduction when considering the number of reachable states for a cube with marked centres will yield an incorrect result.

The six absolute centre cubies for any odd size cube of size greater than 3 behaves exactly the same as those for the size 3 cube. The six sets of four centre cubies in each group for even or odd size cubes of size greater than 4 behave exactly the same as those for the size 4 cube. Furthermore, for odd size cubes all permutations must have even overall parity [12] and in that case the proof is simple. A permutation that would represent odd parity (one or three swaps within a set-of-four centre cubies on a given face) would be unreachable if no change to any other cubies were permitted.

However, because there are many ways of arranging the centre cubies that look the same there is no restriction on the observable arrangements of centre cubies in any given orbit. For example, if two red centre cubies located anywhere within any 24 cubie orbit are swapped no change in observable state results, but such an action without changing anything else would defy parity rules for a cube with marked centres.

The limitation on the possible rotations is often referred to as applying to the *last* set. Use of the word *last* has special relevance if we are manipulating the *last* set-of-four centre cubies in the unscrambling process. However, the factor-of-2 limitation applies to any scrambled arrangement. In such cases *last* really means that if we consider any set-of-four centre cubies and leave the rest of the cube unchanged, the possibilities are limited to an even number of centre cubie swaps (zero or two).

Appendix 3

Higher Resolution Values of Reachable States and Their Logarithms

Figures below for cubes with unmarked centres were extracted from the companion document *cubecalc.xls*. The figures are expressed to 15 significant figures but it is probably safe to say they are correct to 13 significant figures.

Cube size *N*	Number of reachable states *M*	$Y = log_{10}(M)$
2	$3.67416000000000 \times 10^{6}$	6.565158064763500
3	$4.32520032744898 \times 10^{19}$	19.636006227197561
4	$7.40119684156490 \times 10^{45}$	45.869301954819942
5	$2.82870942277742 \times 10^{74}$	74.451588337147406
6	$1.57152858401024 \times 10^{116}$	116.196322284663190
7	$1.95005511837313 \times 10^{160}$	160.290046886884057
8	$3.51737809231095 \times 10^{217}$	217.546219054293243
9	$1.41703923905426 \times 10^{277}$	277.151381876407514
10	$8.29835985127826 \times 10^{349}$	349.918992263710103
11	$1.08540871852024 \times 10^{425}$	425.035593305717776
12	$2.06367789807385 \times 10^{513}$	513.314641912913768
13	$8.76357231708318 \times 10^{603}$	603.942681174814844
14	$5.40963548099816 \times 10^{707}$	707.733168001904239
15	$7.45839677970896 \times 10^{813}$	813.872645483698718
16	$1.49475677548459 \times 10^{933}$	933.174570530681516
17	$6.69092608710526 \times 10^{1054}$	1054.825486232369398
18	$4.35360977202657 \times 10^{1189}$	1189.638849499245599
19	$6.32708138527533 \times 10^{1326}$	1326.801203420826884
20	$1.33661062037299 \times 10^{1477}$	1477.126004907596488
21	$6.30662559561943 \times 10^{1629}$	1629.799797049071176
22	$4.32550437616044 \times 10^{1795}$	1795.636036755734182
23	$6.62623931773829 \times 10^{1963}$	1963.821267117102273
24	$1.47551980674748 \times 10^{2145}$	2145.168945043658683
25	$7.33860690044848 \times 10^{2328}$	2328.865613624920176

The following table provides a comparison of the number of reachable states *M* as calculated in the companion document *cubecalc.xls* and shown in the above table with some published figures. Many references provide accurate figures for size 2, 3, 4 and 5 cubes. That for size 20 is provided by Hardwick [8].

N	Published figures		*cubecalc.xls* figures		Difference	
	M_{pub}		M		$M - M_{pub}$	
	index	*base*	*index*	*base*	*index*	*base*
2	6	3.67416000000000	6	3.67416000000000	0	0.00000000000000
3	19	4.32520032744898	19	4.32520032744898	0	0.00000000000000
4	45	7.40119684156490	45	7.40119684156490	0	0.00000000000000
5	74	2.82870942277742	74	2.82870942277742	0	0.00000000000000
20	1477	1.33661062037297	1477	1.33661062037299	0	0.00000000000002

The previous table indicates that the *cubecalc.xls* Excel calculation of the number of reachable states M is correct to 13 decimal places or 14 significant figures for the size 20 cube. A degradation in the accuracy of the last and possibly the second last digit in some cases is an expected result as the accuracy would degrade during multiplication of numbers limited to 15 significant figures as is the case with Excel.