Note
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AERONAUTICAL RESEARCH LABORATORIES

INSTRUMENT NOTE 63

SINE AND COSINE FUNCTION GENERATORS FOR AN ANALOGUE COMPUTER

by

K. F. Fraser

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SUMMARY

The design of sine and cosine function generators for use in the A.R.L. Analogue Computer is detailed. The generators incorporate diode switching circuits to obtain a piecewise linear approximation to the required functions.

CONTENTS

1. INTRODUCTION

The sine and cosine function generators which are described here were developed at A.R.L. for installation within the A.R.L. Analogue Computer console. Access to the bank of computer amplifiers was allowable for the generation of the required functions within the function generator unit.

2. DESIGN SPECIFICATION

Basically, the required function generators are to have an input voltage which is proportional to an angle (or number) and an output voltage which is proportional to the sine or the cosine of that angle (or number) as the case may be. The following is the design specification to be adhered to.

Functions Required	Sine and cosine (independent).
Angle Coverage	$-\frac{\pi}{2}$ to $+\frac{\pi}{2}$ for both functions.
Accuracy	Better than $\pm 0.5\%$ of maximum value.
Input	50 volt to correspond to 1 radian.
Outputs	$\sin\frac{\pi}{2}$ and $\cos 0$ to correspond to either 50V or 100V as selected by a switch.
Power Supplies	+100V and -100V regulated supplies available in the computer. 6.3V supply for filaments available in the computer.
Operational Amplifiers	Available in the computer.

3. MATHEMATICAL FORMULATION OF DESIGN

(i) Sine Function Generator

The function $y = \sin x$ may be approximated to by a number of linear segments as illustrated in Fig. 1(a). The approximation may be written in the form:

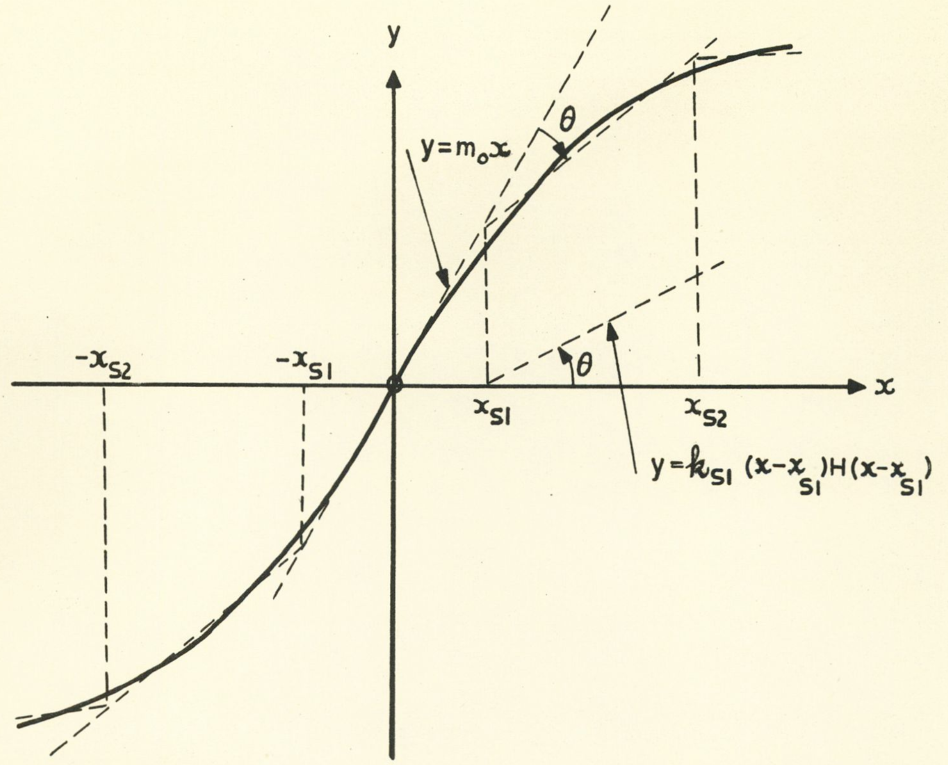


FIG. 1A
APPROXIMATION TO SINE

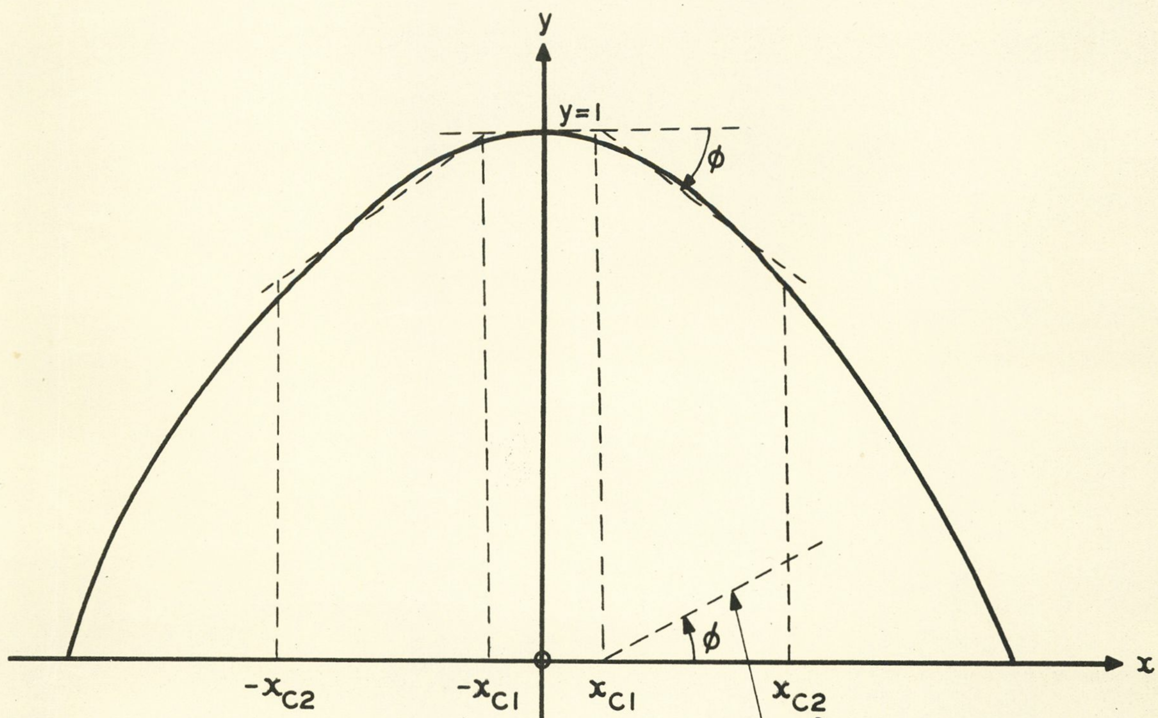


FIG. 1B
APPROXIMATION TO COSINE

$$y = m_o x - \sum_{r=1}^{n} k_{sr}(x - x_{sr})\, H(x - x_{sr}) \qquad (1)$$

$$\text{for } 0 \leqslant x \leqslant +\frac{\pi}{2}$$

where the subscript "s" is used to denote "sine" and

(1) m_o is the gradient of the segment through the origin.

(2) k_{sr} is the change in gradient from the $(r-1)^{th}$ to the r^{th} segment.

(3) x_{sr} is a point of discontinuity of the function and is better known as a "breakpoint".

(4) $H(x - x_{sr})$ is the Heaviside Unit Step Function which has the value unity for $x > x_{sr}$ and zero for $x < x_{sr}$.

In a similar manner it can be seen for the range

$$-\frac{\pi}{2} \leqslant x \leqslant 0$$

$$y = m_o x - \sum_{r=1}^{n} k_{sr}(x + x_{sr})\left[1 - H(x + x_{sr})\right] \quad \ldots \quad (2)$$

(ii) Cosine Function Generator

It can readily be seen by reference to Fig. 1(b) that the approximation to $y = \cos x$ may be written as:

$$y = 1 - \sum_{r=1}^{n} k_{cr}(x - x_{cr})\, H(x - x_{cr})$$

$$\text{for } -\frac{\pi}{2} \leqslant x \leqslant 0 \qquad (3)$$

and

$$y = 1 + \sum_{r=1}^{n} k_{cr}(x + x_{cr})\left[1 - H(x + x_{cr})\right] \qquad (4)$$

$$\text{for } 0 \leqslant x \leqslant +\frac{\pi}{2}$$

where the subscript "c" is used to denote "cosine".

4. ANALOGUE CIRCUITS FOR GENERATING THE DESIRED APPROXIMATION FUNCTIONS

(i) Sine Generator

Consider the simple circuit of Fig. 2.

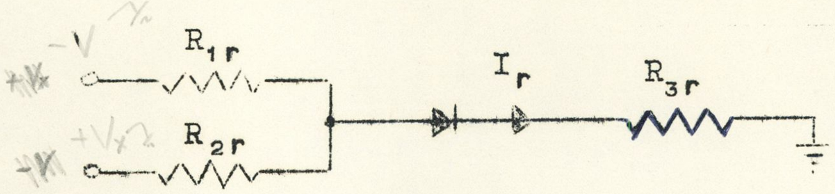


FIG. 2

If it is assumed that the diode acts as a perfect switch, then conduction of the diode will commence when the plate of the diode just reaches zero potential

$$\text{i.e. when } V_x \geqslant \frac{VR_{2r}}{R_{1r}}$$

When the diode is conducting, the equivalent circuit of Fig. 3 applies.

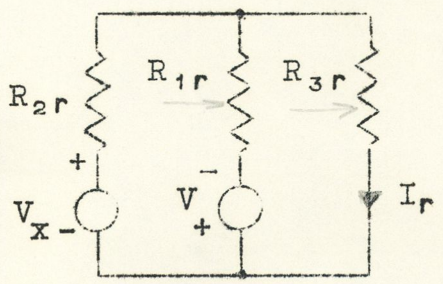


FIG. 3

Simple analysis gives:

$$I_r = \frac{\dfrac{R_{1r}}{R_{1r} + R_{2r}}}{\dfrac{R_{1r} R_{2r}}{R_{1r} + R_{2r}} + R_{3r}} \left(V_x - \frac{R_{2r}}{R_{1r}} V \right) \qquad (5)$$

If it is assumed that the voltage V_x is made proportional to an angle x radian such that

$$V_x = A_1 x .$$

then

$$I_r = \frac{\dfrac{A_1 R_{1r}}{R_{1r}+R_{2r}}}{\dfrac{R_{1r} R_{2r}}{R_{1r}+R_{2r}} + R_{3r}} \left(x - \frac{R_{2r} V}{R_{1r} A_1}\right) \qquad (6)$$

But since conduction occurs only for

$$V_x \geqslant \frac{VR_{2r}}{R_{1r}}$$

or

$$x \geqslant \frac{VR_{2r}}{A_1 R_{1r}}$$

then

$$I_r = \frac{\dfrac{A_1 R_{1r}}{R_{1r}+R_{2r}}}{\dfrac{R_{1r} R_{2r}}{R_{1r}+R_{2r}} + R_{3r}} \left(x - \frac{VR_{2r}}{A_1 R_{1r}}\right) H \left(x - \frac{VR_{2r}}{A_1 R_{1r}}\right) \qquad \ldots (7)$$

It is to be noted that the expression for I_r above is of the same form as the r^{th} term of equation (1).

If in Fig. 2, the diode is reversed and the voltage $-V$ replaced by $+V$, then simple analysis shows that:

$$I_r = \frac{\dfrac{A_1 R_{1r}}{R_{1r}+R_{2r}}}{\dfrac{R_{1r} R_{2r}}{R_{1r}+R_{2r}} + R_{3r}} \left(x + \frac{VR_{2r}}{A_1 R_{1r}}\right) \left[1 - H\left(x + \frac{VR_{2r}}{A_1 R_{1r}}\right)\right] \qquad \ldots (8)$$

This expression for I_r is similar to the r^{th} term of equation (2).

A term of the form $y_o = k_o x$ may be readily obtained by the simple circuit of Fig. 4. Reference to equations (1) and (2) shows that a different sign is required for this term, and hence a $-V_x$ is required.

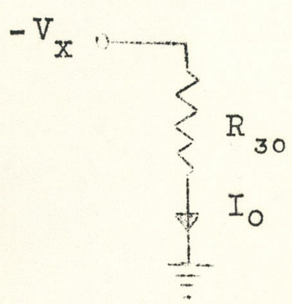


FIG. 4

$$I_o = - \frac{A_1}{R_{30}} x \qquad (9)$$

It is evident that all the terms of equations (1) and (2) may be obtained in the form of currents by using a number of circuits of the type in Fig. 2 and Fig. 4. A summation of the separate currents may be conveniently made at the input of a single operational amplifier to approximate the function in the desired range. An arrangement of the form depicted in Fig. 5 results.

The output of the amplifier is:

$$V_y = \left(I_o - \sum_{r=1}^{n} I_r \right) R_F$$

$$= \frac{A_1 R_F}{R_{30}} x - \frac{\dfrac{A_1 R_{1r} R_F}{R_{1r} + R_{2r}}}{\dfrac{R_{1r} R_{2r}}{R_{1r} + R_{2r}} + R_{3r}} \left(x - \frac{VR_{2r}}{A_1 R_{1r}} \right) H \left(x - \frac{VR_{2r}}{A_1 R_{1r}} \right) \qquad \ldots \quad (10)$$

for $0 \leqslant x \leqslant + \frac{\pi}{2}$

If the voltage V_y is made proportional to a number, y, such that

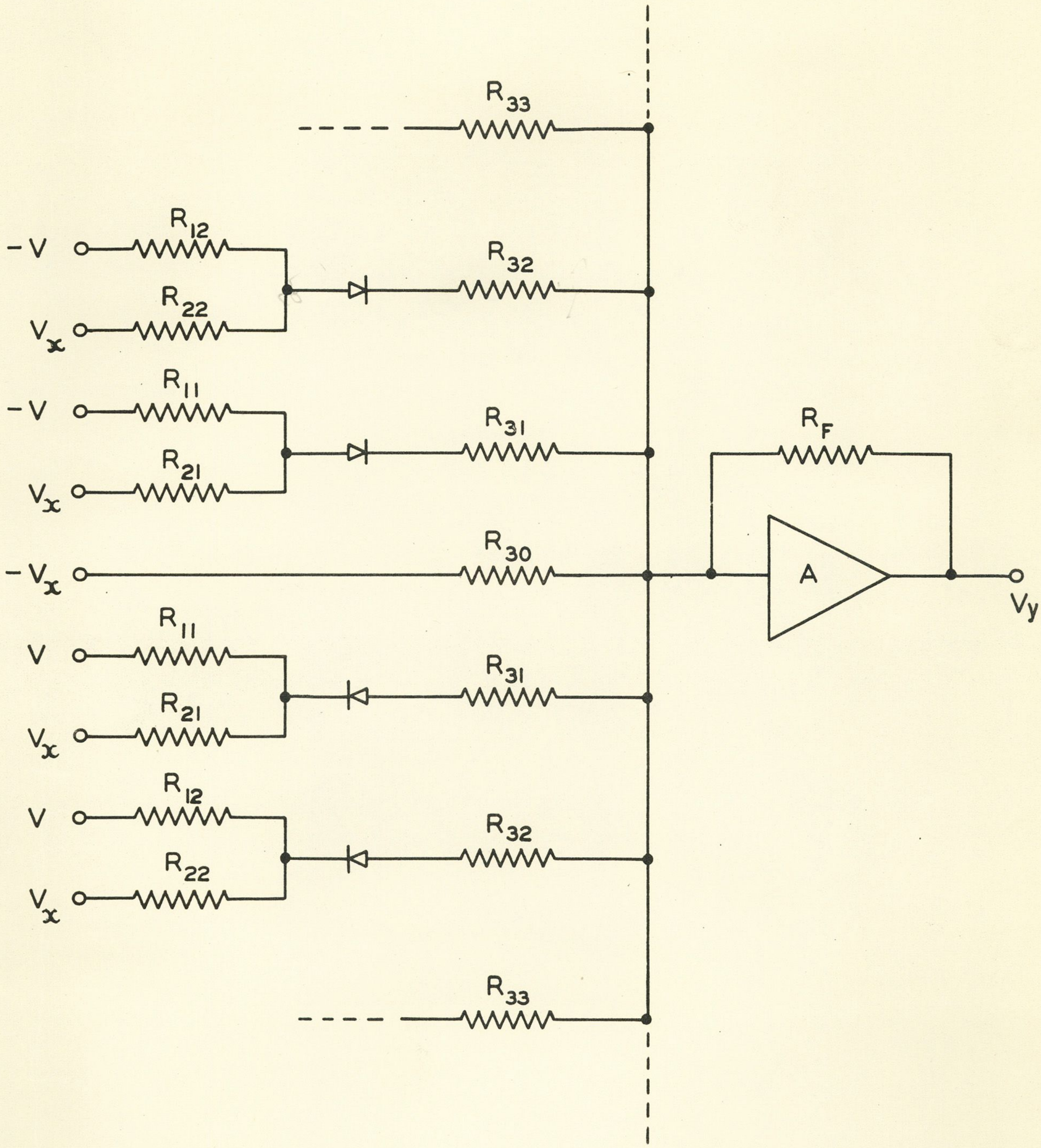
$$V_y = A_2 y$$

then

$$y = \frac{A_1}{A_2} \left[\frac{R_F}{R_{30}} x - \sum_{r=1}^{n} \frac{\dfrac{R_{1r} R_F}{R_{1r} + R_{2r}}}{\dfrac{R_{1r} R_{2r}}{R_{1r} + R_{2r}} + R_{3r}} \left(x - \frac{VR_{2r}}{A_1 R_{1r}} \right) H \left(x - \frac{VR_{2r}}{A_1 R_{1r}} \right) \right] \qquad \ldots \quad (11)$$

for $0 \leqslant x \leqslant + \frac{\pi}{2}$

and similarly for $- \frac{\pi}{2} \leqslant x \leqslant 0$



GENERAL FORM OF CIRCUIT FOR SINE FUNCTION GENERATOR

$$y = \frac{A_1}{A_2}\left[\frac{R_F}{R_{30}}x - \sum_{r=1}^{n} \frac{\dfrac{R_{1r}R_F}{R_{1r}+R_{2r}}}{\dfrac{R_{1r}R_{2r}}{R_{1r}+R_{2r}} + R_{3r}}\left(x + \frac{VR_{2r}}{A_1R_{1r}}\right)\left\{1 - H\left(x + \frac{VR_{2r}}{A_1R_{1r}}\right)\right\}\right] \quad \ldots \quad (12)$$

Comparing (11) and (12) with (1) and (2) we may write

$$\text{BREAKPOINT} \quad x_{sr} = \frac{VR_{2r}}{A_1R_{1r}} \quad (13)$$

$$\text{INCREMENTAL SLOPE} \quad k_{sr} = \frac{\dfrac{A_1}{A_2}\,\dfrac{R_{1r}R_F}{R_{1r}+R_{2r}}}{\dfrac{R_{1r}R_{2r}}{R_{1r}+R_{2r}} + R_{3r}} \quad (14)$$

(ii) Cosine Generator

If a similar analysis is carried out for a cosine generator it can be seen that a circuit of the form depicted in Fig. 6 will produce suitable analogues for the terms of equations (3) and (4). For this circuit, the following equations apply

$$\left.\begin{aligned} V_x &= A_1 x \\ V_y &= A_2 y \end{aligned}\right\} \quad \text{as for sine}$$

$$y = \frac{A_1}{A_2}\left[\frac{R_FV}{R_{30}A_1} + \sum_{r=1}^{n} \frac{\dfrac{R_{1r}R_F}{R_{1r}+R_{2r}}}{\dfrac{R_{1r}R_{2r}}{R_{1r}+R_{2r}} + R_{3r}}\left(x + \frac{VR_{2r}}{A_1R_{1r}}\right)\left\{1 - H\left(x + \frac{VR_{2r}}{A_1R_{1r}}\right)\right\}\right] \quad \ldots \quad (15)$$

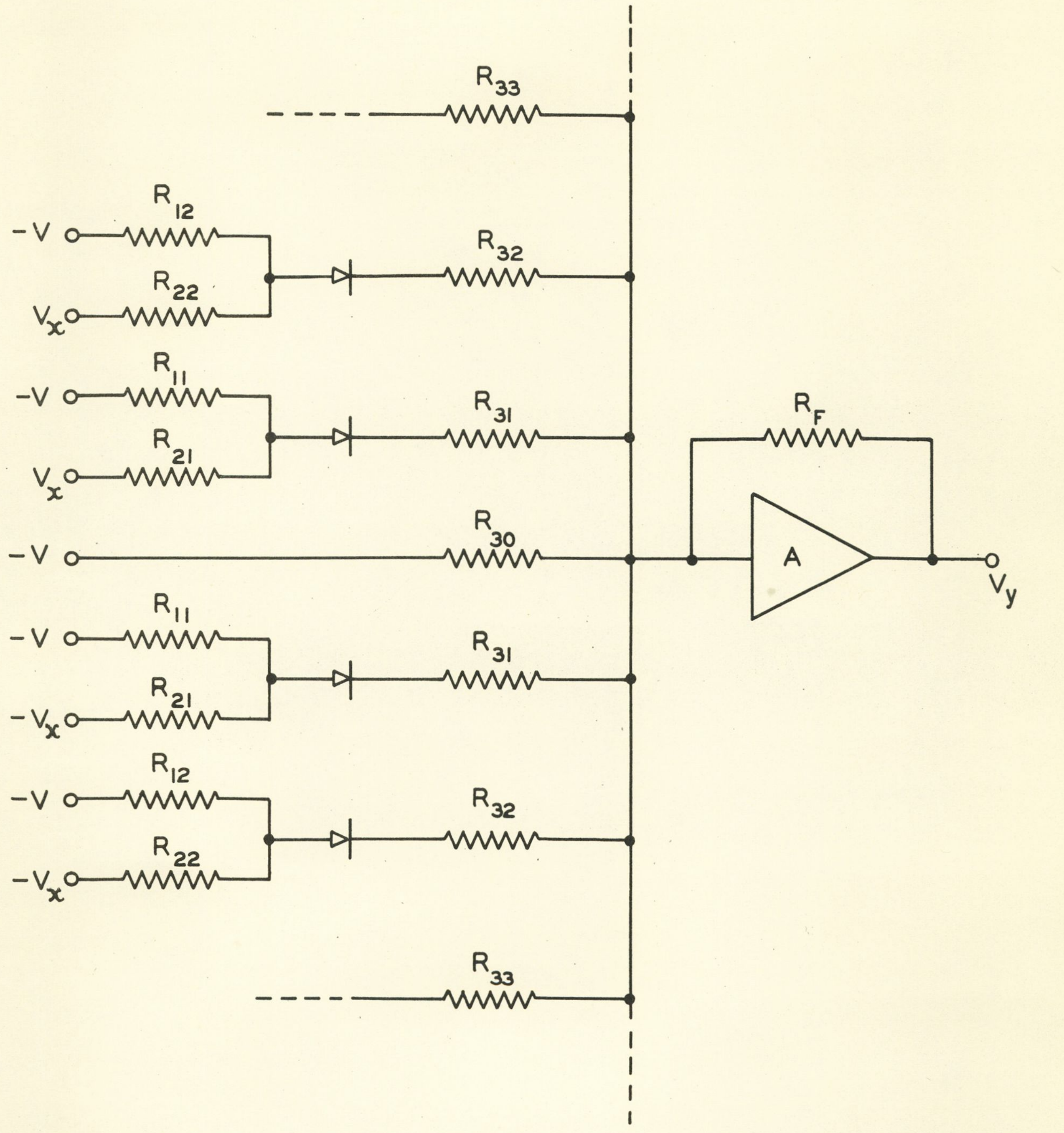
for $0 \leqslant x \leqslant +\frac{\pi}{2}$

$$y = \frac{A_1}{A_2}\left[\frac{R_FV}{R_{30}A_1} \quad \sum_{r=1}^{n} \frac{\dfrac{R_{1r}R_F}{R_{1r}+R_{2r}}}{\dfrac{R_{1r}R_{2r}}{R_{1r}+R_{2r}} + R_{3r}}\left(x - \frac{VR_{2r}}{A_1R_{1r}}\right)H\left(x - \frac{VR_{2r}}{A_1R_{1r}}\right)\right] \quad \ldots \quad (16)$$

for $-\frac{\pi}{2} \leqslant x \leqslant 0$

Note that R_{1r}, R_{2r}, and R_{3r} are not identical

FIG 6



GENERAL FORM OF CIRCUIT FOR COSINE FUNCTION GENERATOR to the resistors of the same notation used for the sine generator.

5. DETERMINATION OF NUMBER OF SEGMENTS REQUIRED

It was found graphically that a sine curve (or a cosine curve) in the range $-\pi/2 \leqslant x \leqslant +\pi/2$ could be approximated to an accuracy of $\pm 0.5\%$ of maximum value by 11 linear segments. The procedure used for the graphical solution can be illustrated with reference to Fig. 7.

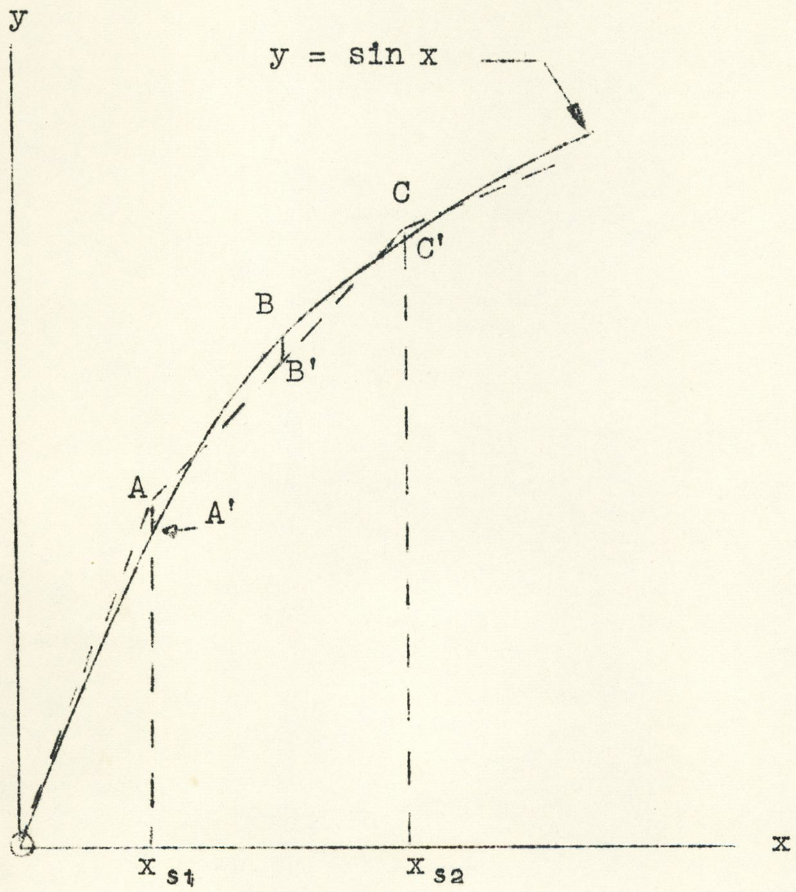


FIG. 7

The slope of the first segment OA was made equal to the slope of $y = \sin x$ at $x = 0$, and extremity A of the first segment was chosen such that AA' was $\leqslant 0.5\%$ of maximum value. The slope of the next segment AD was chosen such that BB' was $\leqslant$ 0.5% of maximum value, and the extremity D was chosen such that DD' was $\leqslant$ 0.5% of maximum value. This process was repeated so as to cover the range $0 \leqslant x \leqslant +\pi/2$. Obviously the piecewise linear approximation for the range $-\pi/2 \leqslant x \leqslant 0$ follows the same pattern. The slope of the segments at $x = \pm \pi/2$ was made equal to the slope of $y = \sin x$ at $x = \pm \pi/2$ (i.e. 0). To provide the 11 requisite segments, 10 breakpoints were obviously required.

The results of the various measurements and computations are tabulated below.

(i) Sine Generator

Breakpoints and Slopes (initially chosen from graphical analysis and later trimmed by actual computations).

SEGMENT BETWEEN BREAK-POINTS	-17° to 17°	17° to 38°	38° to 55°	55° to 71°	71° to 85.2°	85.2° to 90°
SLOPE OF SEGMENT UNIT/DEGREE	0.01745	0.0154	0.0119	0.0080	0.0035	0

Maximum Computed Errors

The maximum errors in the piecewise linear approximation represented by the above table were computed using equations (1) and (2). It is apparent that the angles corresponding to the maximum positive errors will be the breakpoints. The maximum negative errors, which occur between the breakpoints, are approximate only, since they have been computed using integral values of angle only.

ANGLE (DEGREE)	17	29	38	47	55	63	71	78	85.2
ERRORS IN PIECEWISE LINEAR APPROX. TO SINE(% MAX. VALUE)	+0.43	-0.34	+0.44	-0.42	+0.32	-0.47	+0.48	-0.33	+0.35

By specification it was required that 50V correspond to 1 radian. This is equivalent to setting the constant A_1, in the equation $V_x = A_1 x$, equal to 50V/radian.

Hence $1^\circ \sim (0.01745 \times 50)$ volt = 0.8727 volt.

The following table expresses the breakpoints in volt units.

BREAKPOINT SYMBOL	x_{s1}	x_{s2}	x_{s3}	x_{s4}	x_{s5}
BREAKPOINT (DEGREE)	17	38	55	71	85.2
CORRESPONDING BREAKPOINT (VOLT)	14.84	33.16	48.00	61.96	74.35

By specification it was required that 50 volt or 100 volt, as selected by a switch, correspond to 1 unit of y. This is equivalent to setting the constant A_2, in the equation $V_y = A_2 y$, equal to 50 volt/unit or 100 volt/unit respectively. For the case A_2 = 50 volt/unit the following table applies.

Increments in Slope (obtained by subtracting slopes of successive segments).

SLOPE INCREMENT SYMBOL	k_{s1}	k_{s2}	k_{s3}	k_{s4}	k_{s5}
SLOPE INCREMENT UNIT/DEGREE	0.00205	0.0035	0.0039	0.0045	0.0035
SLOPE INCREMENT VOLT/VOLT	0.118	0.201	0.223	0.258	0.201

The slope of the segment at $x = 0$, defined by the symbol m_o in equation (1) is 1 unit/radian (equivalent to 0.01745 unit/degree) or 1 volt/volt.

(ii) Cosine Generator

From a graphical analysis, symmetry considerations show that breakpoints satisfying the accuracy conditions for the cosine curve can be found from the equation:

$$x_{cr} = \frac{\pi}{2} - x_s(6 - r)$$

BREAKPOINT SYMBOL	x_{c1}	x_{c2}	x_{c3}	x_{c4}	x_{c5}
BREAKPOINT (DEGREE)	4.8	19	35	52	73
CORRESPONDING BREAKPOINT (VOLT)	4.20	16.58	30.54	45.38	63.70

The slope increments are, of course, the same magnitude as those for sine, but in the reverse order, and are found from the equation:

$$k_{cr} = k_s(6 - r)$$

SEGMENT BETWEEN BREAKPOINTS	4.8^o to 19^o	19^o to 35^o	35^o to 52^o	52^o to 73^o	73^o to 90^o
SLOPE INCREMENT SYMBOL	k_{c1}	k_{c2}	k_{c3}	k_{c4}	k_{c5}
SLOPE INCREMENT UNIT/DEGREE	0.0035	0.0045	0.0039	0.0035	0.0021
SLOPE INCREMENT VOLT/VOLT	0.201	0.258	0.223	0.201	0.118

The slope of the segment betwepn -4.8^o and 4.8^o is, of course, zero.

6. ACTUAL CIRCUITS EMPLOYED

As mentioned in the specification, regulated supplies +100 Volt and -100 Volt were available in the computer.

i.e. V = 100 Volt

Current drain considerations indicated a requirement of about 250 KΩ total resistance in the voltage dividers formed by R_{1r} and R_{2r}.

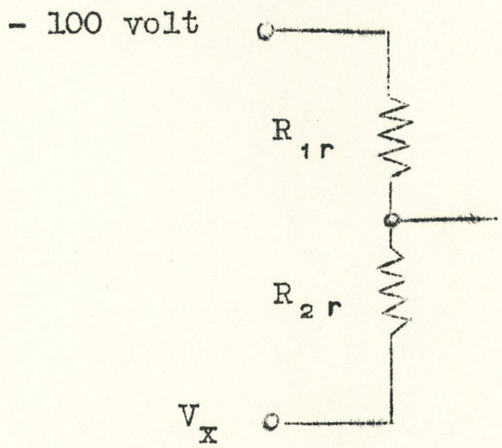


FIG. 8

Referring to the special case of Fig. 8.

If $R_{1r} + R_{2r} = 250\ K\Omega$, and

if x_{sr} is a breakpoint, then using equation (13)

$$R_{1r} = \frac{250}{1 + \frac{A_1 x_{sr}}{100}}\ K\Omega \qquad (17)$$

$$R_{2r} = \frac{250}{1 + \frac{100}{A_1 x_{sr}}}\ K\Omega \qquad (18)$$

where A_1 = 50 volt/radian

Having selected R_{1r} and R_{2r}, R_{3r} may be chosen to provide the required slope from equation (14).

$$R_{3r} = \frac{R_{1r} R_{2r}}{R_{1r} + R_{2r}} \left(\frac{A_1 R_F}{A_2 k_{sr} R_{2r}} - 1 \right) \qquad (19)$$

Potentiometers were added so that breakpoints and slopes could be trimmed to the desired values.

Type 6AL5 twin diode valves are used throughout. The very low reverse leakage and low forward resistance of this diode in the required range makes it ideal for this application. A measured diode characteristic in the range of interest is drawn in Fig. 9.

All fixed resistors except the amplifier feedback resistors are Erie Type 100 (High stability) and all potentiometers are Morganite Type BJ Linear. The Erie resistors are mounted on aluminium strips via polytags and in such a way as to allow proper flow of the circulating air in the computer. The amplifier feedback resistors are precision I.R.C. wire-wound type.

Bulgin switches Type S270 are used for feedback resistor switching. Of 4 switches tested under ambient conditions each had an insulation resistance in excess of 3×10^4 megohm at applied potentials up to 90V.

Computer amplifiers for sign changing at the input (i.e. for obtaining $-V_x$ from V_x) and for current summation are available from the amplifier panel in the computer.

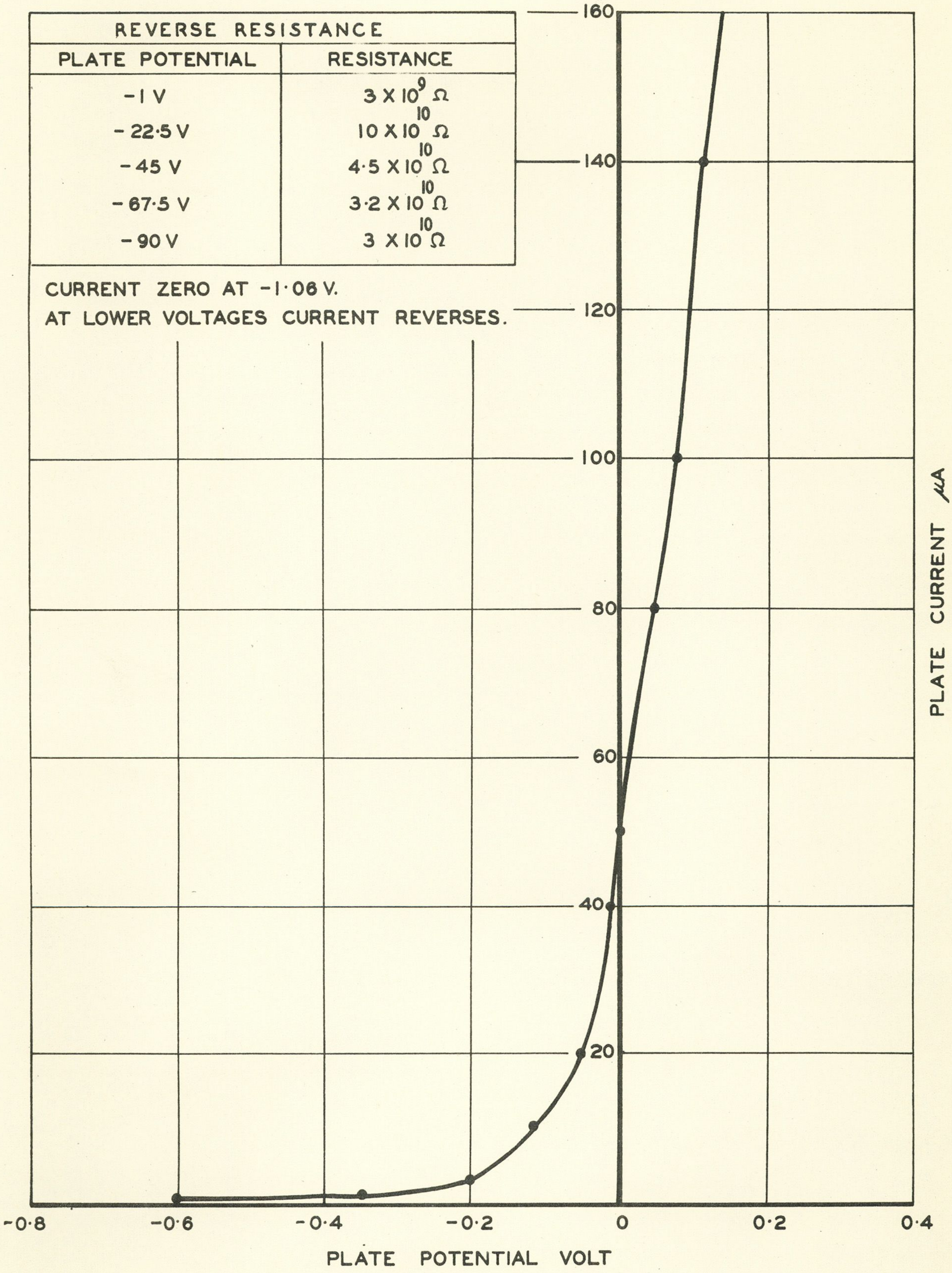
The final circuits together with the associated component values are drawn up in Fig. 10 and Fig. 11.

Noise at full scale positive output for the sine generator (110 volts) was measured as 20 mV peak to peak (i.e. -74 db).

Setting-Up Procedure

Trimming of the actual values of R_{1r}, R_{2r} and R_{3r} to achieve the desired accuracy was carried out using an X-Y recorder. A ramp function was fed to the input of the recorder which plotted output as a function of input. The resulting curve was made to conform within $\pm$ 0.5% to a

FIG. 9



REVERSE RESISTANCE	
PLATE POTENTIAL	RESISTANCE
-1 V	$3 \times 10^{9}\ \Omega$
-22·5 V	$10 \times 10^{10}\ \Omega$
-45 V	$4{\cdot}5 \times 10^{10}\ \Omega$
-67·5 V	$3{\cdot}2 \times 10^{10}\ \Omega$
-90 V	$3 \times 10^{10}\ \Omega$

CHARACTERISTIC OF 6AL5 (ONE HALF ONLY)

true sine curve by successive adjustments and readjustments of potentiometers 1C, 1D etc. (Refer to Fig. 10 and Fig. 11).

Drawings

The following drawings relate to the unit described.

Chassis Details Front and Rear	A.R.L. DRG. No. 4282
Panel Engraving	A.R.L. DRG. No. 4293
Circuit Details	A.R.L. DRG. No. 4593

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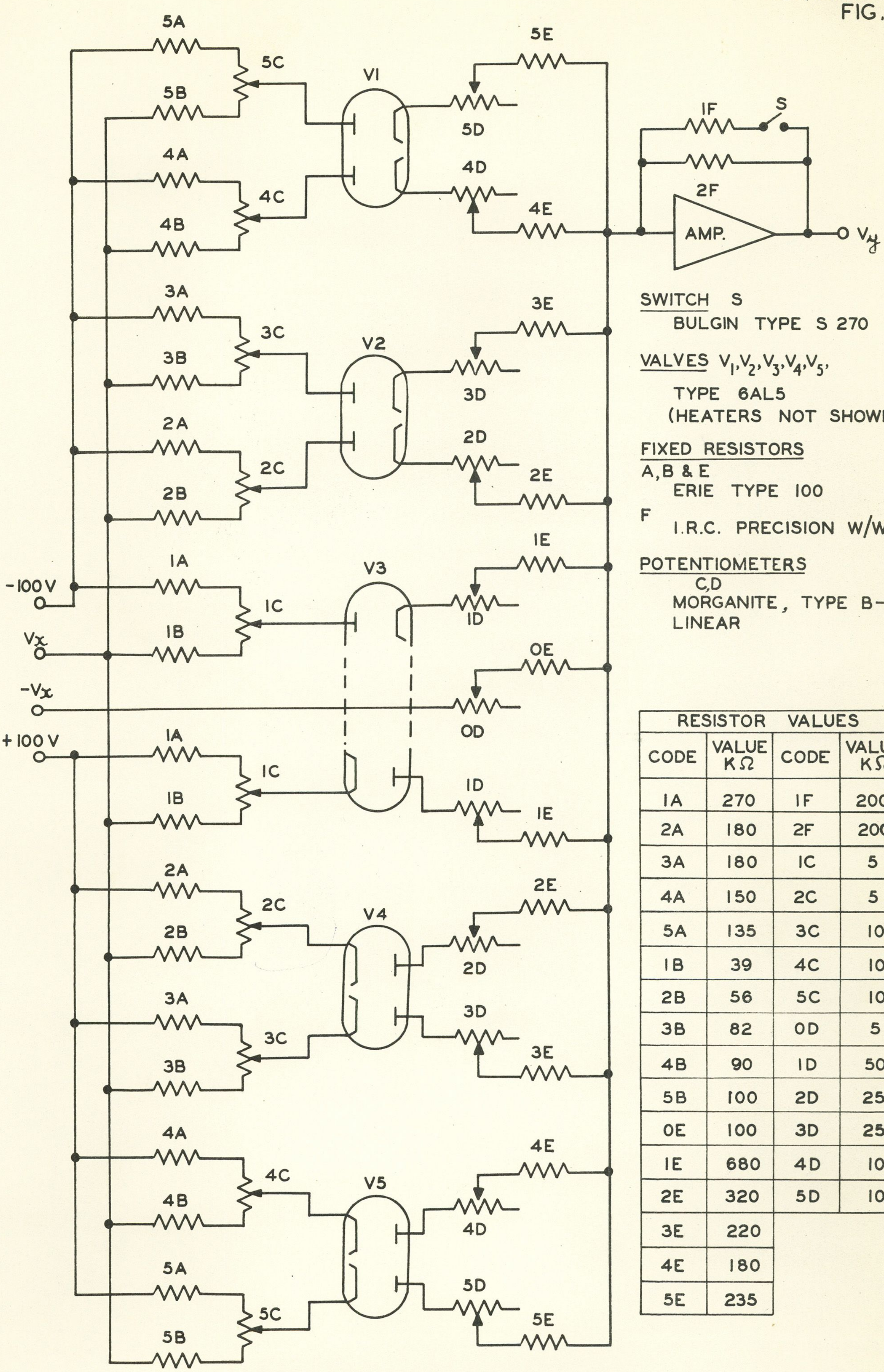


SWITCH S
BULGIN TYPE S 270

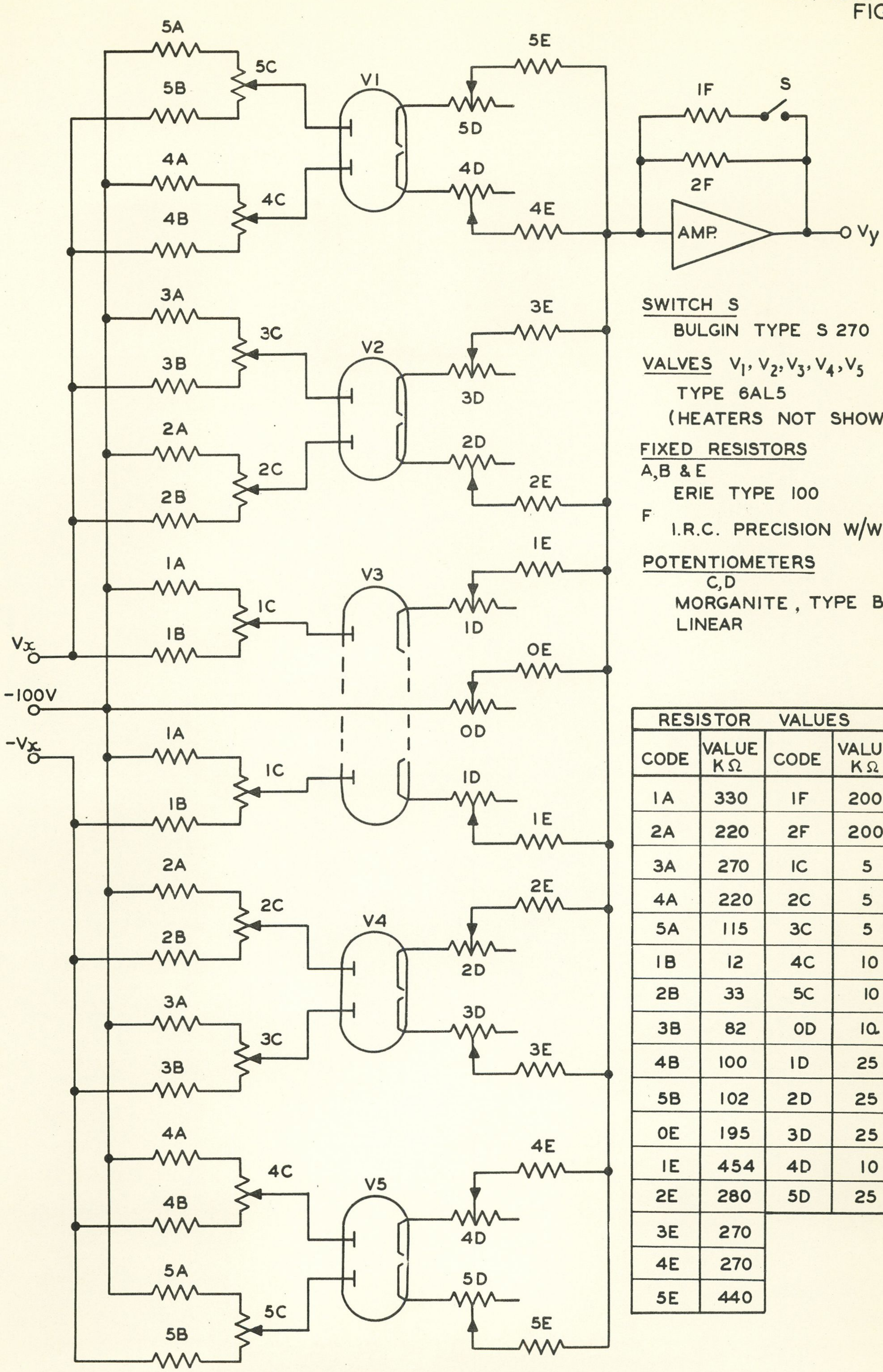
VALVES V_1, V_2, V_3, V_4, V_5,
TYPE 6AL5
(HEATERS NOT SHOWN)

FIXED RESISTORS
A, B & E
ERIE TYPE 100
F
I.R.C. PRECISION W/W

POTENTIOMETERS
C, D
MORGANITE, TYPE B-J
LINEAR

RESISTOR VALUES			
CODE	VALUE KΩ	CODE	VALUE KΩ
1A	270	1F	200
2A	180	2F	200
3A	180	1C	5
4A	150	2C	5
5A	135	3C	10
1B	39	4C	10
2B	56	5C	10
3B	82	0D	5
4B	90	1D	50
5B	100	2D	25
0E	100	3D	25
1E	680	4D	10
2E	320	5D	10
3E	220		
4E	180		
5E	235		

SINE GENERATOR - ACTUAL CIRCUIT



SWITCH S
BULGIN TYPE S 270

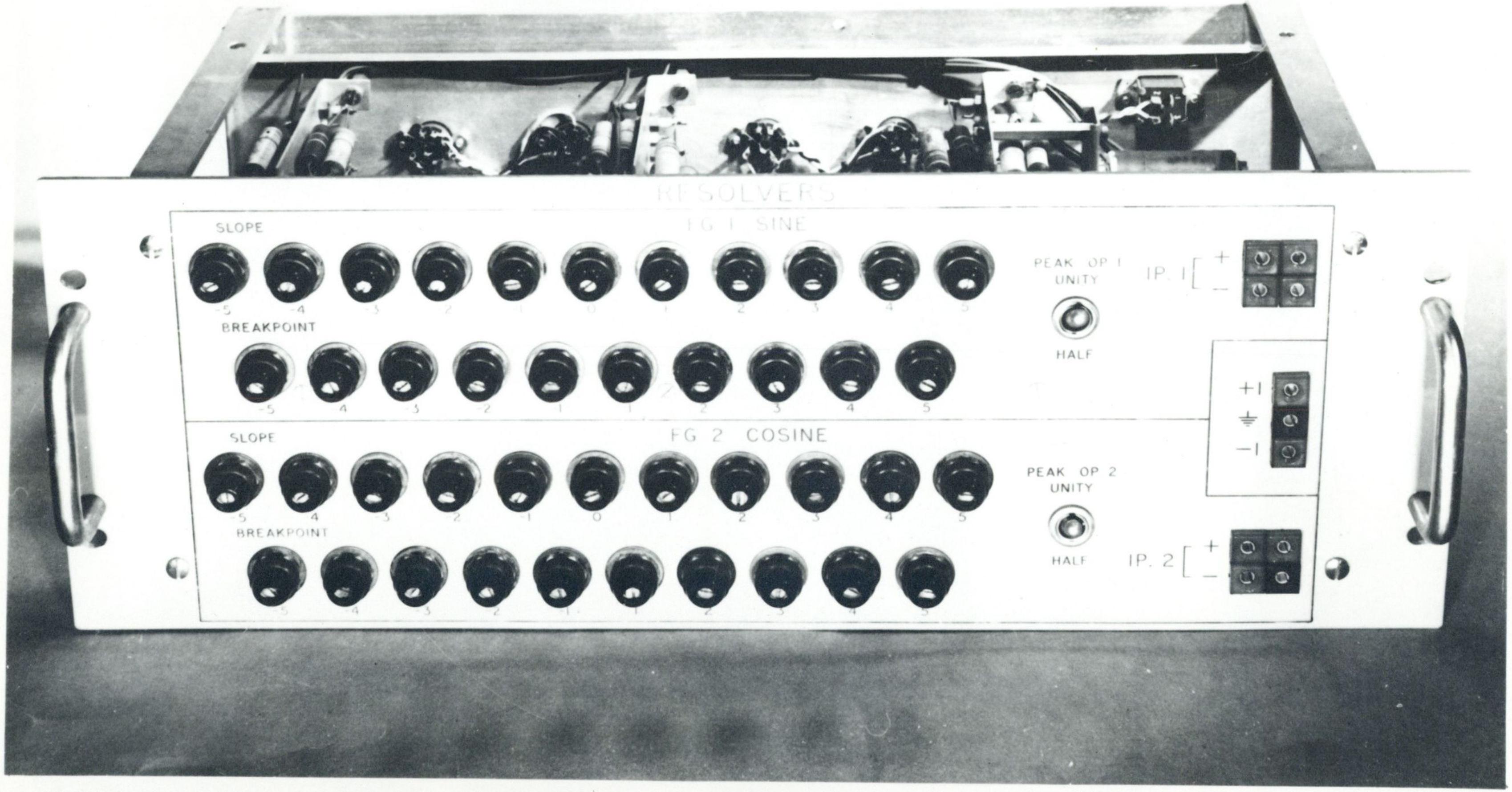
VALVES V_1, V_2, V_3, V_4, V_5
TYPE 6AL5
(HEATERS NOT SHOWN)

FIXED RESISTORS
A, B & E
ERIE TYPE 100
F
I.R.C. PRECISION W/W

POTENTIOMETERS
C, D
MORGANITE, TYPE B-J
LINEAR

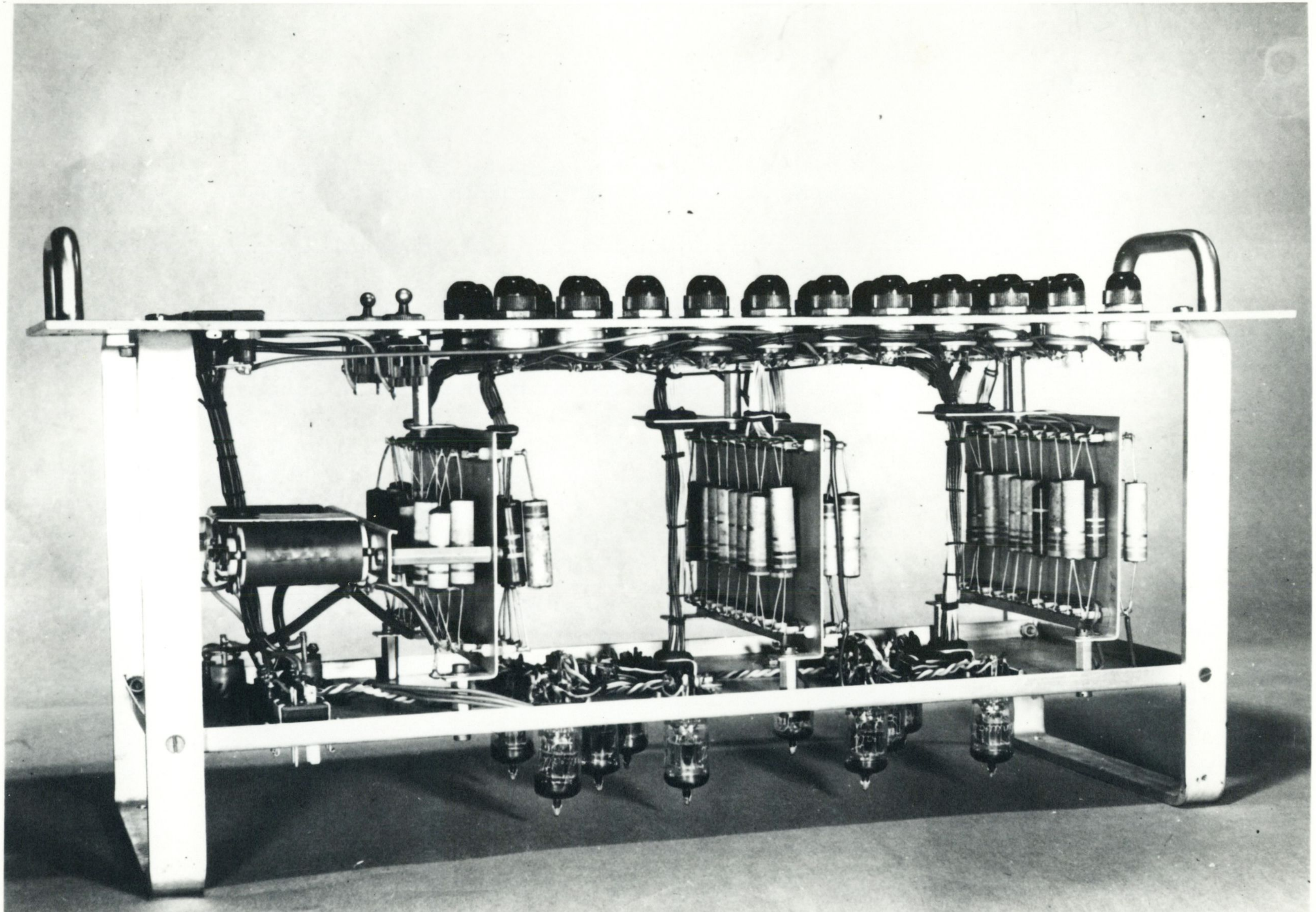
RESISTOR	VALUES		
CODE	VALUE KΩ	CODE	VALUE KΩ
1A	330	1F	200
2A	220	2F	200
3A	270	1C	5
4A	220	2C	5
5A	115	3C	5
1B	12	4C	10
2B	33	5C	10
3B	82	0D	10
4B	100	1D	25
5B	102	2D	25
0E	195	3D	25
1E	454	4D	10
2E	280	5D	25
3E	270		
4E	270		
5E	440		

COSINE GENERATOR - ACTUAL CIRCUIT



FRONT VIEW OF FUNCTION GENERATORS

INST. NOTE 63
FIG. 12



SIDE VIEW OF FUNCTION GENERATORS

INST. NOTE 63
FIG. 13