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SOME ERRORS IN A SIMPLIFIED COMPUTATION OF THE MALKARA TRAJECTORY

by

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December 1967

MELBOURNE

PRINTED BY DEPARTMENT OF SUPPLY, CENTRAL DRAWING OFFICE, MARIBYRNONG, VICTORIA 3032

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SUMMARY

A simplified determination of Malkara trajectory in which motion of the missile in the vertical plane is considered to be essentially dependent only on the missile pitch control demand signal, may, in cases where a simultaneous yaw manoeuvre is added, lead to substantial errors in trajectory slope estimation. Moreover, if the missile angle of attack is not considered in the programme fed to the pitch control system, an additional error will arise. In this article a more precise analysis is used to calculate some of these errors for an extreme case. General expressions in matrix form have been derived throughout for the solution of the attitude and flight path of an airborne body.

CONTENTS

NOTATION

$ox_M y_M z_M$	Orthogonal set of axes fixed within the missile.
$oA_R A_P A_S$	Set of axes associated with the gyroscope, not necessarily orthogonal.
$oxyz$	Orthogonal set of axes corresponding to the initial (launch) attitude of the $ox_M y_M z_M$ axes. These axes are fixed relative to the earth.
$ox_W y_W z_W$	Orthogonal set of axes within the missile which define the direction of motion of the missile (relative wind is directed along ox_W).
$ox_F y_F z_F$	Orthogonal set of axes fixed relative to the earth. oz_F is vertical.
p	Rate of rotation of missile about missile roll axis (ox_M).
q	Rate of rotation of missile about missile pitch axis (oy_M).
r	Rate of rotation of missile about missile yaw axis (oz_M).
φ_G	Rotation of missile about gyroscope roll gimbal axis (oA_R).
θ_G	Rotation of missile about gyroscope pitch gimbal axis (oA_P).
ψ_G	Rotation of missile about gyroscope rotor spin axis (oA_S).
α	Angle of attack of the missile.
β	Angle of sideslip of missile.
φ_o	Angle of inclination of launch vehicle.
θ_Q	Angle of launch of missile.
θ_T	Angle representing instantaneous inclination of trajectory with respect to earth.
ψ_T	Yaw angle of trajectory with respect to $Ox_F y_F z_F$ axes.
ℓ_1, ℓ_2, ℓ_3	Direction cosines of trajectory with respect to $ox_F y_F z_F$ axes.
θ_D	Instantaneous inclination of the "desired" (see below) trajectory with respect to the initial launch attitude.
ψ_D	Instantaneous yaw angle (measured in the horizontal plane) of the "desired" trajectory with respect to the initial launch attitude.
θ_E	Trajectory inclination error arising in the "simplified" (see below) analysis. $\theta_E = \theta_T - (\theta_D + \theta_Q)$.
ϵ	That part of the trajectory inclination error which arises because the gyroscope rotor spin axis is not vertical.

NOTATION (cont'd)

$\underline{v_T}$	Velocity of the missile (relative to earth).
$\underline{\Omega}$	Instantaneous angular velocity of the missile.
t	Time elapsed since moment of launch.
s	Distance measured along the trajectory.
h	Distance measured along the projection of the trajectory on the horizontal plane.

Vector Notation. A bar above the symbol as in $\bar{x}_M$ defines the unit vector along ox_M.

A bar below the symbol as in $\underline{\Omega}$ defines a vector of magnitude Ω.

DEFINITION OF TERMS

"Desired" Trajectory. A theoretical flight path which if followed would line the missile exactly on the target. The "desired" trajectory forms the basis for the control signals fed to the missile.

"True" Trajectory. The flight path predicted by the more exact analysis of this paper if the missile is fed with control signals based on the "desired" trajectory.

"Simplified" Analysis. Assumes that the actual trajectory corresponds to the "desired" trajectory.

1. INTRODUCTION

A reasonably precise computation of the Malkara trajectory, when the missile is subject to manoeuvres about both pitch and yaw axes simultaneously, presents a most formidable problem. In fact, to combine the missile aerodynamics with the flight programme in anything but a very simplified form, would rarely be justified. For this reason a desired trajectory, based on known performance characteristics of the Malkara, was initially specified. Trajectory slope errors, resulting from the use of simplified pitch and yaw programmes based on this desired trajectory, were computed by comparison with a more precisely computed trajectory.

2. RELATIONSHIP BETWEEN TIME DERIVATIVES OF MISSILE AND GYRO ROTATIONS

2.1 Particulars of Malkara Gyroscope

Define a set of Cartesian axes $ox_My_Mz_M$ fixed within the missile as depicted in Fig. 1 where:

ox_M is the missile roll axis
oy_M is the missile pitch axis
oz_M is the missile yaw axis.

Now the Malkara is fitted with a single vertical gyroscope (i.e. rotor spin axis initially along oz_M), the outer gimbal axis of which is fixed within the missile and coincides at all times with the missile pitch axis. The pitch potentiometer is mounted on the outer gimbal axis and the roll potentiometer is mounted on the inner gimbal axis. In the caged position (the position in which the rotor is run up to speed) the rotor spin axis coincides with the missile yaw axis. At this stage it will be useful to introduce a set of gyro axes $oA_RA_PA_S$ where:

oA_R is the inner or roll gimbal axis
oA_P is the outer or pitch gimbal axis
oA_S is the rotor spin axis.

It is to be noted that the above set of axes does not, except in the caged position, form an orthogonal set, as oA_S is not perpendicular to oA_P. Fig. 2 illustrates how the missile can rotate with respect to the gyro.

2.2 Matrix Relating Unit Vectors Along Missile Axes to Unit Vectors Along Gyro Axes

In the rotor caged position the gyro axes and the missile axes coincide to form an orthogonal set oxyz, say. Define θ_G and φ_G as the rotations of the missile about the pitch and roll gimbal axes as read by the respective potentiometers, and further, assume these angles are zero for the caged rotor position. To completely define a new missile attitude a further angle needs to be specified. It is convenient to define a rotation ψ_G of the missile about the rotor spin axis of the gyro. Now assume that the missile attitude changes from oxyz to another arbitrary attitude $ox_My_Mz_M$. If the following finite rotations are performed in the order given:

(1) Rotation ψ_G about z axis to form ox'y'z'

(2) Rotation φ_G about x' axis to form ox"y"z"

(3) Rotation θ_G about y" axis to form $ox_My_Mz_M$

then φ_G and θ_G will correspond to the actual gyro readings. ψ_G of course is not read by this gyro. If the order of rotations is changed then that set of angles will not correspond to the gyro readings. Fig. 3 depicts the various finite rotations defined above, and also the relation between the missile and gyro axes.

If the notation $\bar{x}_M$ is used to define the unit vector along the x_M axis, and likewise for the other axes, then from Fig. 3 it can readily be seen that:

$$\begin{bmatrix} \bar{A}_R \\ \bar{A}_P \\ \bar{z}'' \end{bmatrix} = \begin{bmatrix} \cos\theta_G & 0 & \sin\theta_G \\ 0 & 1 & 0 \\ -\sin\theta_G & 0 & \cos\theta_G \end{bmatrix} \begin{bmatrix} \bar{x}_M \\ \bar{y}_M \\ \bar{z}_M \end{bmatrix} \quad ..(1)$$

$$\begin{bmatrix} \bar{A}_R \\ \bar{A}_P \\ \bar{A}_S \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \sin\varphi_G & \cos\varphi_G \end{bmatrix} \begin{bmatrix} \bar{A}_R \\ \bar{A}_P \\ \bar{z}'' \end{bmatrix} \quad ..(2)$$

$$\begin{bmatrix} \overline{A}_R \\ \overline{A}_P \\ \overline{A}_s \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \sin\varphi_G & \cos\varphi_G \end{bmatrix} \begin{bmatrix} \cos\theta_G & 0 & \sin\theta_G \\ 0 & 1 & 0 \\ -\sin\theta_G & 0 & \cos\theta_G \end{bmatrix} \begin{bmatrix} \overline{x}_M \\ \overline{y}_M \\ \overline{z}_M \end{bmatrix}$$

$$\begin{bmatrix} \overline{A}_R \\ \overline{A}_P \\ \overline{A}_s \end{bmatrix} = \begin{bmatrix} \cos\theta_G & 0 & \sin\theta_G \\ 0 & 1 & 0 \\ -\cos\varphi_G \sin\theta_G & \sin\varphi_G & \cos\varphi_G \cos\theta_G \end{bmatrix} \begin{bmatrix} \overline{x}_M \\ \overline{y}_M \\ \overline{z}_M \end{bmatrix} \quad .. (3)$$

Matrix equation (3) relates the unit vectors along the missile axes to the unit vectors along the gyro axes.

2.3 Time Derivatives of Rotations

For the following analysis it will be assumed that the missile is turning with an instantaneous angular velocity $\underline{\Omega}$ (where the bar beneath the symbol signifies a vector). Define p, q and r as rates of rotation of the missile about the missile axes ox_M, oy_M and oz_M respectively. It follows that

$$\underline{\Omega} = \frac{d\varphi_G}{dt}\overline{A}_R + \frac{d\theta_G}{dt}\overline{A}_P + \frac{d\psi_G}{dt}\overline{A}_s \quad .. (4)$$

$$\underline{\Omega} = p\,\overline{x}_M + q\,\overline{y}_M + r\,\overline{z}_M \quad .. (5)$$

Note that since $\overline{A}_R$, $\overline{A}_P$ and $\overline{A}_s$ are not all mutually perpendicular (except in the caged attitude) then the resolute of $\underline{\Omega}$ in the direction of $\overline{A}_R$ (i.e. $\underline{\Omega} \cdot \overline{A}_R$) is not $\frac{d\varphi_G}{dt}$, the component of $\underline{\Omega}$ in the direction of $\overline{A}_R$.

Combining equations (4) and (5) and writing in matrix form we obtain:

$$\begin{bmatrix} \frac{d\varphi_G}{dt} & \frac{d\theta_G}{dt} & \frac{d\psi_G}{dt} \end{bmatrix} \begin{bmatrix} \overline{A}_R \\ \overline{A}_P \\ \overline{A}_s \end{bmatrix} = \begin{bmatrix} p & q & r \end{bmatrix} \begin{bmatrix} \overline{x}_M \\ \overline{y}_M \\ \overline{z}_M \end{bmatrix} \quad .. (6)$$

Using equation (3)

$$\begin{bmatrix} \frac{d\varphi_G}{dt} & \frac{d\theta_G}{dt} & \frac{d\psi_G}{dt} \end{bmatrix} \begin{bmatrix} \cos\theta_G & 0 & \sin\theta_G \\ 0 & 1 & 0 \\ -\cos\varphi_G \sin\theta_G & \sin\varphi_G & \cos\varphi_G \cos\theta_G \end{bmatrix} = \begin{bmatrix} p & q & r \end{bmatrix} .. (7)$$

Transposing both sides

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \cos\theta_G & o & -\cos\varphi_G \sin\theta_G \\ o & 1 & \sin\varphi_G \\ \sin\theta_G & o & \cos\varphi_G \cos\theta_G \end{bmatrix} \begin{bmatrix} \frac{d\varphi_G}{dt} \\ \frac{d\theta_G}{dt} \\ \frac{d\psi_G}{dt} \end{bmatrix} \qquad ..(8)$$

Now the Malkara is roll stabilized by means of a control system which maintains $\varphi_G = o$. Hence for stabilization assumed perfect

$$\varphi_G = o \quad \text{and} \quad \frac{d\varphi_G}{dt} = o.$$

Under these conditions equation (8) simplifies to:

$$p = -\frac{d\psi_G}{dt} \sin\theta_G \qquad ..(9)$$

$$q = \frac{d\theta_G}{dt} \qquad ..(10)$$

$$r = \frac{d\psi_G}{dt} \cos\theta_G \qquad ..(11)$$

It is interesting to note from these equations that if a yaw rotation occurs (i.e. $r \neq o$ and $\frac{d\psi_G}{dt} \neq o$) then if $\theta_G \neq o$ $p \neq o$ and hence rotation will occur about the missile roll axis.

3. SPECIFICATION OF MISSILE ATTITUDE IN TERMS OF GYRO ANGLES

The attitude of the missile relative to its initial attitude (defined earlier by the orthogonal set of axes oxyz) can be expressed in terms of the three angles ψ_G, φ_G and θ_G. In the following analysis the unit vector relationship between the oxyz and the o x_M y_M z_M frames will be calculated.

Referring to Fig. 3 we can write:

(i) For rotation ψ_G about z axis to form ox'y'z'

$$\begin{bmatrix} \bar{x}' \\ \bar{y}' \\ \bar{z}' \end{bmatrix} = \begin{bmatrix} \cos\psi_G & \sin\psi_G & o \\ -\sin\psi_G & \cos\psi_G & o \\ o & o & 1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} \qquad ..(12)$$

(ii) For rotation φ_G about x' to form ox"y"z".

$$\begin{bmatrix} \bar{x}'' \\ \bar{y}'' \\ \bar{z}'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_G & \sin\varphi_G \\ 0 & -\sin\varphi_G & \cos\varphi_G \end{bmatrix} \begin{bmatrix} \bar{x}' \\ \bar{y}' \\ \bar{z}' \end{bmatrix} \quad ..(13)$$

(iii) For rotation of θ_G about y" to form $ox_My_Mz_M$

$$\begin{bmatrix} \bar{x}_M \\ \bar{y}_M \\ \bar{z}_M \end{bmatrix} = \begin{bmatrix} \cos\theta_G & 0 & -\sin\theta_G \\ 0 & 1 & 0 \\ \sin\theta_G & 0 & \cos\theta_G \end{bmatrix} \begin{bmatrix} \bar{x}'' \\ \bar{y}'' \\ \bar{z}'' \end{bmatrix} \quad ..(14)$$

Hence

$$\begin{bmatrix} \bar{x}_M \\ \bar{y}_M \\ \bar{z}_M \end{bmatrix} = \begin{bmatrix} \cos\theta_G & 0 & -\sin\theta_G \\ 0 & 1 & 0 \\ \sin\theta_G & 0 & \cos\theta_G \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_G & \sin\varphi_G \\ 0 & \sin\varphi_G & \cos\varphi_G \end{bmatrix} \begin{bmatrix} \cos\psi_G & \sin\psi_G & 0 \\ -\sin\psi_G & \cos\psi_G & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix}$$

$$\begin{bmatrix} \bar{x}_M \\ \bar{y}_M \\ \bar{z}_M \end{bmatrix} = \begin{bmatrix} \cos\theta_G \cos\psi_G - \sin\theta_G \sin\varphi_G \sin\psi_G & \cos\theta_G \sin\psi_G + \sin\theta_G \sin\varphi_G \cos\psi_G & -\sin\theta_G \cos\psi_G \\ -\cos\varphi_G \sin\psi_G & \cos\varphi_G \cos\psi_G & \sin\varphi_G \\ \sin\theta_G \cos\psi_G + \cos\theta_G \sin\varphi_G \sin\psi_G & \sin\theta_G \sin\psi_G - \cos\theta_G \sin\varphi_G \cos\psi_G & \cos\theta_G \cos\varphi_G \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} \quad ..(15)$$

For roll stabilization $\varphi_G = 0$ and equation (15) simplifies to:

$$\begin{bmatrix} \bar{x}_M \\ \bar{y}_M \\ \bar{z}_M \end{bmatrix} = \begin{bmatrix} \cos\theta_G \cos\psi_G & \cos\theta_G \sin\psi_G & -\sin\theta_G \\ -\sin\psi_G & \cos\psi_G & 0 \\ \sin\theta_G \cos\psi_G & \sin\theta_G \sin\psi_G & \cos\theta_G \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} \quad ..(16)$$

4. RELATION OF FLIGHT PATH ATTITUDE TO MISSILE ATTITUDE

The direction of the relative air may be related to the missile axes ($ox_My_Mz_M$) in terms of two angles, α and β, as illustrated in Fig. 4. For the case of no wind, as will be assumed in this paper, the resultant velocity of the missile relative to the ground, $\underline{v_T}$, will be minus the relative air velocity (the apparent air velocity as seen from the missile) and will be directed along ox_W as shown in Fig. 4. The set of axes $ox_Wy_Wz_W$ may be obtained from the missile axes $ox_My_Mz_M$ by two successive rotations, α about y_M followed by β about the z_W (or new z_M) axis.

α, normally referred to as the angle of attack of the missile, is the inclination of the longitudinal centre-line (ox_M axis) of the missile with respect to the projection of the relative air velocity vector ($\overline{v_T}$ directed along ox_W) onto the missile "vertical" plane of symmetry ($ox_M z_M$ plane).

β, normally referred to as the sideslip angle of the missile, is the inclination of the relative air velocity vector to the missile "vertical" plane of symmetry ($ox_M z_M$ plane).

From Fig. 4 we may write:

$$\begin{bmatrix} \bar{x}_W \\ \bar{y}_W \\ \bar{z}_W \end{bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta & o \\ -\sin\beta & \cos\beta & o \\ o & o & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & o & -\sin\alpha \\ o & 1 & o \\ \sin\alpha & o & \cos\alpha \end{bmatrix} \begin{bmatrix} \bar{x}_M \\ \bar{y}_M \\ \bar{z}_M \end{bmatrix}$$

$$\begin{bmatrix} \bar{x}_W \\ \bar{y}_W \\ \bar{z}_W \end{bmatrix} = \begin{bmatrix} \cos\alpha\ \cos\beta & \sin\beta & -\sin\alpha\ \cos\beta \\ \cos\alpha\ \sin\beta & \cos\beta & \sin\alpha\ \sin\beta \\ \sin\alpha & o & \cos\alpha \end{bmatrix} \begin{bmatrix} \bar{x}_M \\ \bar{y}_M \\ \bar{z}_M \end{bmatrix} \qquad ..(17)$$

Since α and β are usually quite small the following approximations may be made:

$$\cos\beta \simeq 1$$
$$\sin\beta \simeq \beta$$
$$\cos\alpha \simeq 1$$
$$\sin\alpha \simeq \alpha$$
$$\sin\alpha\ \sin\beta \simeq o$$

Using the above approximations equation (17) simplifies to

$$\begin{bmatrix} \bar{x}_W \\ \bar{y}_W \\ \bar{z}_W \end{bmatrix} = \begin{bmatrix} 1 & \beta & -\alpha \\ -\beta & 1 & o \\ \alpha & o & 1 \end{bmatrix} \begin{bmatrix} \bar{x}_M \\ \bar{y}_M \\ \bar{z}_M \end{bmatrix} \qquad ..(18)$$

5. SPECIFICATION OF MISSILE LAUNCH ATTITUDE

It is convenient to relate the missile launch attitude to a set of earth axes $ox_F\ y_F\ z_F$ for which z_F is vertical. Assume that the launcher vehicle is resting on a slope of φ_o as depicted in Fig. 5.

The case in which the launcher arm is set at 0^o to the vehicle as regards yaw will be considered. The effect of the initial roll φ_o will be greatest under these conditions.

The launch attitude may be obtained from the set of earth axes $ox_F\ y_F\ z_F$ by two rotations; a rotation of φ_o about x_F followed by a rotation θ_Q about y in that order. Writing down the direction cosine matrices we obtain:

$$\begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} = \begin{bmatrix} \cos\theta_Q & o & -\sin\theta_Q \\ o & 1 & o \\ \sin\theta_Q & o & \cos\theta_Q \end{bmatrix} \begin{bmatrix} 1 & o & o \\ o & \cos\varphi_o & \sin\varphi_o \\ o & -\sin\varphi_o & \cos\varphi_o \end{bmatrix} \begin{bmatrix} \bar{x}_F \\ \bar{y}_F \\ \bar{z}_F \end{bmatrix}$$

$$\begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} = \begin{bmatrix} \cos\theta_Q & \sin\theta_Q \sin\varphi_o & -\sin\theta_Q \cos\varphi_o \\ o & \cos\varphi_o & \sin\varphi_o \\ \sin\theta_Q & -\cos\theta_Q \sin\varphi_o & \cos\theta_Q \cos\varphi_o \end{bmatrix} \begin{bmatrix} \bar{x}_F \\ \bar{y}_F \\ \bar{z}_F \end{bmatrix} \quad ..(19)$$

6. RELATION OF FLIGHT PATH ATTITUDE TO EARTH FRAME

It is convenient to specify the flight path attitude in terms of two angles, namely, ψ_T, the angle through which the tangent to the projection of the flight path on a horizontal plane rotates from its initial position (parallel to x_F) to the point in question and θ_T, the inclination of the trajectory to the horizontal at the point in question.

If two successive rotations, ψ_T about the z_F axis, and θ_T about the new y_F axis are carried out in that order then the x_W axis will be obtained in the final set. These rotations are depicted in Fig. 6.

It is readily seen that

$$\bar{x}_W = \begin{bmatrix} \cos\psi_T \cos\theta_T & \sin\psi_T \cos\theta_T & -\sin\theta_T \end{bmatrix} \begin{bmatrix} \bar{x}_F \\ \bar{y}_F \\ \bar{z}_F \end{bmatrix} \quad ..(20)$$

$$\bar{x}_W = \begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \end{bmatrix} \begin{bmatrix} \bar{x}_F \\ \bar{y}_F \\ \bar{z}_F \end{bmatrix} \quad ..(21)$$

say

where

$$\tan\psi_T = \frac{\ell_2}{\ell_1} \quad ..(22)$$

$$\sin\theta_T = -\ell_3 \quad ..(23)$$

Using equations (19),(18) and (16) the direction cosines ℓ_1, ℓ_2 and ℓ_3 may be obtained in terms of φ_o, θ_Q, ψ_G, θ_G, α and β.

$$\begin{bmatrix} \bar{x}_W \\ \bar{y}_W \\ \bar{z}_W \end{bmatrix} = \begin{bmatrix} 1 & \beta & -\alpha \\ -\beta & 1 & o \\ \alpha & o & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_G \cos\psi_G & \cos\theta_G \sin\psi_G & -\sin\theta_G \\ -\sin\psi_G & \cos\psi_G & o \\ \sin\theta_G \cos\psi_G & \sin\theta_G \sin\psi_G & \cos\theta_G \end{bmatrix} \begin{bmatrix} \cos\theta_Q & \sin\theta_Q \sin\varphi_o & -\sin\theta_Q \cos\varphi_o \\ o & \cos\varphi_o & \sin\varphi_o \\ \sin\theta_Q & -\cos\theta_Q \sin\varphi_o & \cos\theta_Q \cos\varphi_o \end{bmatrix} \begin{bmatrix} \bar{x}_F \\ \bar{y}_F \\ \bar{z}_F \end{bmatrix} \qquad ..(24)$$

And hence

$$\ell_1 = \cos\theta_Q \cos\theta_G \cos\psi_G - \sin\theta_Q \sin\theta_G - \beta \cos\theta_Q \sin\psi_G - \alpha \left\{\cos\theta_Q \sin\theta_G \cos\psi_G + \sin\theta_Q \cos\theta_G\right\} \qquad ..(25)$$

$$\ell_2 = \sin\theta_Q \sin\varphi_o \cos\theta_G \cos\psi_G + \cos\varphi_o \cos\theta_G \sin\psi_G + \cos\theta_Q \sin\varphi_o \sin\theta_G \qquad ..(26)$$

$$+ \beta(\cos\varphi_o \cos\psi_G - \sin\theta_Q \sin\varphi_o \sin\psi_G) - \alpha(\sin\theta_Q \sin\varphi_o \sin\theta_G \cos\psi_G + \cos\varphi_o \sin\theta_G \sin\psi_G - \cos\theta_Q \sin\varphi_o \cos\theta_G).$$

$$\ell_3 = -\sin\theta_Q \cos\varphi_o \cos\theta_G \cos\psi_G + \sin\varphi_o \cos\theta_G \sin\psi_G - \cos\theta_Q \cos\varphi_o \sin\theta_G \qquad ..(27)$$

$$+ \beta(\sin\theta_Q \cos\varphi_o \sin\psi_G + \sin\varphi_o \cos\psi_G) + \alpha(\sin\theta_Q \cos\varphi_o \sin\theta_G \cos\psi_G - \sin\varphi_o \sin\theta_G \sin\psi_G - \cos\theta_Q \cos\varphi_o \cos\theta_G)$$

7. METHOD OF FINDING SOLUTION TO THE ACTUAL TRAJECTORY

The position of the missile at any instant may be specified in terms of a set of Cartesian co-ordinates (x_F, y_F, z_F) relating to the $ox_Fy_Fz_F$ set of axes as defined in Sec. 6.

From equations (23) and (27) of Sec. 6 θ_T may be expressed as a function of the parameters φ_o, θ_Q, ψ_G, θ_G, α and β. For any particular trajectory φ_o and θ_Q will be constants and ψ_G, θ_G, α and β will be functions of time. Hence if all the above parameters are known θ_T may be expressed as a function of time. Similarly, using equations (22), (25) and (26), ψ_T may be expressed as a function of the same parameters, and hence also as a function of time.

In Fig. 7 a generalised trajectory has been drawn and the projection of this trajectory on the ox_Fy_F plane (a horizontal plane) is shown. An incremental displacement of the missile at some arbitrary point W on the trajectory is broken up into components in the directions of the ox_F, oy_F and oz_F axes. To relate these components the following need to be defined:

s is the distance measured along the trajectory.

ℓ is the distance measured along the projection of the trajectory on the horizontal plane.

θ_T is the instantaneous inclination of the trajectory with respect to the horizontal (Refer to Sec. 6).

ψ_T is the instantaneous angle between the projection of the trajectory on the horizontal plane and the ox_F axis ($\tan\psi_T = \frac{dy_F}{dx_F}$). For equivalent definition of ψ_T refer to Sec. 6.

$\underline{v_T}$ (magnitude v_T) is the resultant velocity of the missile and is directed along the tangent to the trajectory.

From Fig. 7 it can readily be seen that the following relationships apply:

$$d\ell = ds\cos\theta_T$$
$$dz_F = -ds\sin\theta_T$$
$$dx_F = d\ell\cos\psi_T$$
$$dy_F = d\ell\sin\psi_T$$

Dividing the above equations by dt and using the relationship $v_T = \frac{ds}{dt}$ we obtain:

$$\frac{dx_F}{dt} = \frac{ds}{dt} \cos \theta_T \cos \psi_T$$
$$= v_T \cos \theta_T \cos \psi_T \qquad ..(28)$$

$$\frac{dy_F}{dt} = \frac{ds}{dt} \cos \theta_T \sin \psi_T$$
$$= v_T \cos \theta_T \sin \psi_T \qquad ..(29)$$

$$\frac{dz_F}{dt} = -\frac{ds}{dt} \sin \theta_T$$
$$= - v_T \sin \theta_T \qquad ..(30)$$

Integrating equations (28), (29) and (30) with respect to time the co-ordinates of any point on the trajectory may be expressed as a function of time. Time t is the time elapsed since the moment of launch.

$$x_F = \int_0^t v_T \cos \theta_T \cos \psi_T \, dt \qquad ..(31)$$

$$y_F = \int_0^t v_T \cos \theta_T \sin \psi_T \, dt \qquad ..(32)$$

$$z_F = -\int_0^t v_T \sin \theta_T \, dt \qquad ..(33)$$

In the following section no attempt will be made to determine the co-ordinates of points on the trajectory. Computations will be confined to the determination of the trajectory slope θ_T, as a function of time for a particular case.

8. DETERMINATION OF TRAJECTORY SLOPE ERRORS ARISING IN A SIMPLIFIED DETERMINATION OF MALKARA TRAJECTORY

8.1 Outline of Method Used in the Error Calculations

Steps 1 to 5 detailed below indicate the method used for the computation of trajectory slope errors arising in a simplified determination of a particular Malkara trajectory (path of the missile after launch).

Step 1. The locations of the launcher (L), the aimer (A) and the slope angle (γ_F of Fig. 8) of a line joining the aimer and the target (T) positions will be specified. It will be assumed that, prior to reaching the target, the missile is brought into a straight path along the AT line for the "desired" trajectory (defined in Step 2).

Step 2. A trajectory will be specified in conjunction with the L, A and T positions. If this trajectory is followed exactly, the missile will eventually line up exactly on the AT line without any correction being applied by the aimer. This trajectory, so defined, will be referred to as the "desired trajectory". The specification for the desired trajectory will be based on known performance characteristics of the Malkara but, in practice, it may not be possible to programme the Malkara to follow such a flight path.

The desired trajectory may be further defined in terms of the "desired path of the missile in pitch" and the "desired path of the missile in yaw." The desired path of the missile pitch is a plot of the vertical height of the missile (in direction oz_F of Fig. 7) as a function of distance (p of Fig. 7) along the projection of the desired trajectory on the ox_Fy_F plane. The desired path of the missile in yaw is a plot of the projection of the desired trajectory on the ox_Fy_F plane (corresponds to curve oT' of Fig. 7 drawn in the ox_Fy_F plane).

Step 3. The slope angle of the desired trajectory (with reference to the initial launch attitude), defined as θ_D, will be determined as a function of time with the aid of some Malkara performance data. Although θ_D is defined in the same manner as θ_T (except that θ_T is measured with respect to the horizontal plane whereas θ_D is measured with respect to the initial launch attitude, similar to θ_G), it has been re-defined here to distinguish between the slope angle of the "desired" and the "true" trajectory (defined in Step 4). Similarly ψ_D, which bears the same similarity to ψ_T as θ_D does to θ_T, will be determined as a function of time.

Step 4. The slope angle θ_T of the "true" trajectory will be determined as a function of time. The "true" trajectory is defined as the trajectory predicted by the analyses of the previous sections if the pitch control signal receives a demand angle equal to θ_D (or $\theta_D + \alpha$ when the missile angle of attack is taken account of) and if it assumed that the rate of yaw rotation r is equal to $\frac{d\psi_D}{dt}$. Note that there is no closed loop control of the yaw rotation. The "true" trajectory will also be subject to error because of simplifying assumptions but will be sufficiently accurate to enable approximate computation of departures of trajectory slope from those given for the desired trajectory. A "simplified" computation predicts that the true slope will be the same as the desired slope.

(Inverted commas, as used in association with the word "true" in "true trajectory," will be retained in the text because of the ficticious nature of the word as used there).

Step 5. The "true" trajectory slope angle θ_T obtained as outlined in Step 4 will be compared with $\theta_Q + \theta_D$. The difference will be a measure of the slope angle errors which arise in the simplified analysis.

8.2 Desired Path of the Missile in Pitch

At the time when this determination was made the desired path of the missile in pitch included as an invariant, the maximum height of the missile trajectory above the AT line (actually 61 foot perpendicular to the AT line approximately). Before firing of the missile the launch angle was adjusted so that the maximum height of the missile above the AT line would be as specified above. The specification of an invariant enables a single valued solution for the desired slope angle to be obtained (essential for programming the pitch control signal fed to the missile).

It appeared that under certain conditions the use of the invariant defined above could result in insufficient terrain clearance. A proposal was therefore put forward in which the missile elevation at launch (γ_1 of Fig. 8) was made invariant and of a value such as to provide sufficient terrain clearance in all likely cases. A special case of this latter proposal forms the basis for the desired path of the missile in pitch to be used in the error calculation. In Fig. 8 the desired path of the missile in pitch has been drawn.

The following parameters marked in Fig. 8 are defined:

γ_1 is the angle of launch of the missile (equivalent to θ_Q in Sec. 5).

γ_F is the inclination of a line through the aimer and target positions.

h is the vertical height of the aimer above the launcher.

h_o is the maximum perpendicular distance between the desired trajectory and a line drawn through the aimer and target positions.

a is the horizontal separation between the aimer and the target.

b is the length of the initial linear portion of the trajectory.

The path of the missile in pitch can be broadly divided into four sections (Refer to Fig. 8).

(i) An initial linear portion (contains boost phase).
(ii) A circular portion of radius r_1.
(iii) A circular portion of radius r_2 corresponding to the pull-out.
(iv) A straight portion directed along the aimer-target line.

The limits below apply for h, a and γ_F:

$$-50 \text{ ft.} < h < 50 \text{ ft.}$$
$$3 \text{ yd.} < a < 200 \text{ yd.}$$
$$-10^o < \gamma_F < 10^o.$$

For the specific case under computation assume

$$\gamma_1 = 20^o$$
$$\gamma_F = -10^o$$
$$h = 50 \text{ ft.}$$
$$a = 300 \text{ ft.}$$
$$b = 300 \text{ ft.}$$

r_1 = 1950 ft. (which corresponds to the highest rate of turn in the pitch plane for the Malkara at standard air density)

$$r_2 = 6000 \text{ ft.}$$

The expression below for h_o has been derived in the Appendix with the aid of Fig. 9.

$$h_o = a \sin \gamma_F - h \cos \gamma_F + b \sin(\gamma_1 - \gamma_F) + r_1 \left[1 - \cos(\gamma_1 - \gamma_F)\right] \quad ..(34)$$

Substitution of the values shown above in Equation 34 gives h_o = 310 foot.

From the desired path of the missile in pitch drawn in Fig.8 we may write:

$$h_o = r_1 + r_2 - (r_1 + r_2) \cos \eta$$

$$= (r_1 + r_2) (1 - \cos \eta)$$

$$\cos \eta = 1 - \frac{h_o}{r_1 + r_2} \qquad ..(35)$$

Hence in this case

$$\eta = 16.05^o \simeq 16^o$$

The missile boost takes place during the initial linear portion of the flight path. A typical boost consists of an acceleration of 20g (assumed constant) for 0.67 second. Under these conditions the missile would acquire a speed of 430 ft./ sec. and would have moved a distance of 144 ft. at the end of the boost. It is assumed that the speed acquired at the end of boost remains constant throughout the remainder of the missile flight.

From the previous paragraph it can be seen that 156 ft. of linear motion of the missile follows after the end of the boost till point B (Fig. 8) is reached. Hence the time taken for this portion will be 0.36 sec..

Rotation at constant angular velocity will occur between points B and P of Fig. 8. For r_1 = 650 yd. = 1950 ft. and assuming speed remains constant at 430 ft./sec., the angular velocity will be 12.6°/ sec.. Similarly between points P and Q (r_2 = 6000 ft.) the angular velocity will be 4.1°/sec.. Hence the time to go from B to P will be 3.65 sec. (total angle $\gamma_1 - \gamma_F + \eta$ is 46°) and from P to Q will be 3.90 sec. (η = 16°).

Using the results calculated above θ_D, defined as the instantaneous inclination (measured in the vertical plane) of the desired trajectory with respect to the initial launch attitude, may be readily plotted as a function of time as shown in Fig. 10.

8.3 Desired Path of the Missile in Yaw

The desired path of the missile in yaw consists of a straight portion followed by a circular portion to bring the missile onto the AT line (Refer to Fig. 11). The missile is assumed to yaw at its maximum rate of turn during the circular portion. For horizontal flight at st ndard air density this rate is 11. °/sec.. The rate of turn (in this special case) in the horizontal plane is assumed, for simplicity, to be

11.6°/sec.. Hence the rate of turn used here may be somewhat unrealistic. However a "desired" trajectory suitable for error computations is still defined.

If it is assumed for this special case that the missile ceases to rotate in yaw 4.69 sec. after launch then δ (as indicated in Fig. 11) $= 42.5^{\circ}$. Note that the time elapsed till the missile reaches point P in the pitch trajectory of Fig. 8 (i.e. 4.69 sec.) does not have to be the same as the time elapsed till the missile ceases to rotate in yaw, but is merely made equal here for convenience.

Define ψ_D as the yaw angle of Fig. 11 measured relative to the initial launch attitude. ψ_D is plotted as a function of time in Fig. 12.

8.4 Slope Angle of "True" Trajectory

According to equations (23) and (27) the angles φ_o, θ_Q, α, β, θ_G and ψ_G are required for the calculation of the slope angle of the "true" trajectory (For definition of "true" trajectory refer to Step 4 of Sec. 8.1).

8.4.1 Specification of θ_Q and φ_o

The launch attitude is specified in terms of the $ox_Fy_Fz_F$ frame and the angles θ_Q (angle of launch of the missile) and φ_o (angle of inclination of the launch vehicle). For this determination assume $\varphi_o = o$ and put $\theta_Q = \gamma_1 = 20^{\circ}$ (Refer to Sec. 8.1).

8.4.2 Determination of α and β

For a steady state turn in pitch it has been found from aerodynamic considerations that the following relationship applies approximately for Malkara:

$$\alpha = -0.574\, q \quad -0.0403 \quad \text{radian}$$

(Note that the convention for positive α as defined in Sec. 4 is opposite to that which is used in many texts).

..(36)

$$\alpha = -0.574\, q \quad - 2.3 \quad \text{degrees}$$

where q is measured in degrees/sec.

From equation (10) we have:

$$q = \frac{d\theta_G}{dt}$$

Assuming that θ_G is made equal to θ_D (Refer to Sec. 8.4.3) then, with the aid of Fig. 10 and equation (36), α may be tabulated as a function of time as shown below.

t sec.	0 - 1.03	1.03 - 4.69	4.69 - 8.59	8.59 -
q deg./sec.	0	- 12.6	4.1	0
α deg. rad.	- 2.3 - 0.040	4.9 0.086	- 4.6 - 0.080	- 2.3 - 0.040

Equation (36) is strictly valid only for steady state conditions and hence the values of α shown in the above table are liable to be in considerable error in the regions of abrupt change of q (i.e. higher order derivatives exist at these times).

For a steady state turn in yaw it has been found from aerodynamic considerations that the following relationship applies approximately for Malkara.

$$\beta = 0.574 \; r \qquad ..(37)$$

Assuming $r = \frac{d\psi_D}{dt}$ (Refer to Sec. 8.4.4) and neglecting first and higher order derivatives of r, β may be tabulated as a function of time with the aid of Fig. 12 and equation (37).

t sec.	0 - 1.03	1.03 - 4.69	4.69 -
r deg./sec.	0	11.6	0
β deg. rad.	0 0	6.5 0.116	0 0

8.4.3 Determination of θ_G

For an ideal pitch control system (employing a high gain error sensing amplifier) the angle θ_G as indicated on the gyro pitch gimbal is equal to the demand angle fed to the system. Assuming ideal operation of the control system and assuming the demand angle fed to the system is made equal to θ_D, then $\theta_G = \theta_D$.

(If the missile angle of attack α were taken into account in the pitch control then we would put $\theta_G = \theta_D - \alpha$).

Values of θ_G as a function of time may be read from Fig. 10.

8.4.4 Determination of ψ_G

For this determination assume that the rate of rotation about the missile yaw axis is made equal to $\frac{d\psi_D}{dt}$

$$\text{i.e.} \quad r = \frac{d\psi_D}{dt}$$

It is to be understood that, without the aid of closed loop control of the missile yaw angle, it would be difficult to arrange for the missile yaw rotation to be equal to ψ_D (an arbitrarily chosen angle which is a function of time as indicated in Fig. 12). However, even though the missile yaw rotation for $r = \frac{d\psi_D}{dt}$ may be an impractical one, it may still be used for calculating the order of the trajectory slope angle errors (defined in Step 5 of Sec. 8.1).to be expected in similar practical cases.

Putting $r = \frac{d\psi_D}{dt}$ and with the aid of Fig. 12 we may write

$$r = 11.6 \left[H(t - 1.03) - H(t - 4.69) \right] \text{ deg./sec.} \qquad ..(38)$$

where H (t - k) is the Heaviside Unit Step Function. It has the value o for $t < k$ and the value 1 for $t > k$.

Recalling equation (11)

$$\left(r = \frac{d\psi_G}{dt} \cos \theta_G \right)$$

we may write

$$\psi_G = \int_0^t r \sec \theta_G \, dt \qquad ..(39)$$

This integration is allowable since $\frac{d\psi_G}{dt}$ is fixed in direction in space [along the gyro rotor spin axis] and r has been made an independent function.

For the range $o < t < 1.03$ sec. $\psi_G = 0$

For the range $1.03 \text{ sec.} < t < 4.69$ sec.

$$\psi_G = 11.6 \int_{1.03}^{t} \sec \theta_G \, dt \quad \text{deg.}$$

$$\text{and} \quad \theta_G = -12.6 \; (t - 1.03) \text{ deg.}$$

$$\text{Hence} \quad \psi_G \text{ (deg.)} = 11.6 \int_{1.03}^{t} \sec \left[12.6 \; (t - 1.03) \right] dt$$

$$\psi_G \text{ (rad)} = \frac{11.6}{12.6} \ln \left[\sec \left[12.6 (t - 1.03) \right] + \tan \left[12.6 (t - 1.03) \right] \right]$$

$$\psi_G \text{ (deg)} = 52.8 \ln \left\{ \sec \left[12.6 (t - 1.03) \right] + \tan \left[12.6 (t - 1.03) \right] \right\} \quad ..(40)$$

For $t > 4.69$ $r = o$ and ψ_G remains constant at its value at 4.69 sec.

Using equation (40) ψ_G may be plotted as a function of time as shown in Fig. 13.

8.4.5 Determination of θ_T

Substituting $\varphi_o = 0$ in equation (27) and using equation (23) we obtain:

$$\begin{aligned} \sin \theta_T = \; & \sin \theta_Q \cos \theta_G \cos \psi_G \\ & + \cos \theta_Q \sin \theta_G \\ & - \beta \sin \theta_Q \sin \psi_G \\ & + \alpha (\cos \theta_Q \cos \theta_G - \sin \theta_Q \sin\theta_G \cos\psi_G) \quad ..(41) \end{aligned}$$

where θ_T is the slope angle of the "true" trajectory.

All variables on the R.H.S. of equation (41) are known as functions of time and hence θ_T may be calculated as a function of time.

8.5 Trajectory Slope Errors

In the simplified determination it is assumed that the actual trajectory slope angle is given by θ_D. Define the trajectory slope angle error as the difference between the "true" trajectory slope angle and the desired trajectory slope angle $\theta_D + \theta_Q$ (referred to the ox_Fy_F plane).

$$\theta_E = \theta_T - (\theta_Q + \theta_D) \quad ..(42)$$

If we make $\theta_G = \theta_D$ then

$$\theta_E = \theta_T - (\theta_Q + \theta_G)$$

Equation (41) may be re-arranged to enable θ_E to be expressed explicitly. The third term on the R.H.S. of equation (41) contains β as a coefficient and hence will be zero when the missile ceases to rotate in yaw (i.e. when $\frac{d\psi_G}{dt} = o$) for zero wind conditions. Hence this term does not contribute here to the final steady state slope error.

For $\beta = 0$ we may write using equation (41):

$$\sin\theta_T = \cos\theta_Q(\sin\theta_G + \alpha\cos\theta_G) + \sin\theta_Q\cos\psi_G(\cos\theta_G - \alpha\sin\theta_G)$$

$$\simeq \cos\theta_Q\sin(\theta_G + \alpha) + \sin\theta_Q\cos\psi_G\cos(\theta_G + \alpha)$$

since α is small

$$\sin\theta_T = \sin(\theta_Q + \theta_G + \alpha) - (1 - \cos\psi_G)\sin\theta_Q\cos(\theta_G + \alpha)$$

$$\simeq \sin(\theta_Q + \theta_G + \alpha - \epsilon) \quad ..(43)$$

$$\text{where } \epsilon = \text{arsin}\ \frac{(1 - \cos\psi_G)\sin\theta_Q\cos(\theta_G + \alpha)}{\cos(\theta_Q + \theta_G + \alpha)} \quad ..(44)$$

and ϵ is assumed to be small.

Hence $\theta_T = \theta_Q + \theta_G + \alpha - \epsilon$

and $\theta_E = \alpha - \epsilon$ (Putting $\theta_G = \theta_D$ in equation (42))
for $\beta = 0$

It is to be noted that ϵ is a function of α, but α only slightly modifies the final steady state value of ϵ.

It is of interest to note the trajectory slope errors for two special cases.

(1) $\theta_Q = 0$ (Launch attitude horizontal for which gyroscope rotor spin axis is vertical).

In this case $\epsilon = 0$ (See equation (44))
and $\theta_E = \alpha$

Errors in trajectory slope are therefore due entirely to the neglect of α in the programme fed to the pitch control system.

If α is taken into account in the programme fed to the pitch control system such that $\theta_G = \theta_D - \alpha$, then $\theta_E = 0$ and no slope errors are predicted.

It is to be emphasised that yaw rotation, ψ_G, does not give rise to any errors for this special case of the rotor spin axis vertical.

(2) $\psi_G = 0$ (No turning in yaw)

From equation (44) $\epsilon = 0$ and hence $\theta_E = \alpha$. Once again the trajectory slope errors are due to the neglect of α in the programme fed to the pitch control system. If α is taken into account in the programme fed to the pitch control system then no slope errors are predicted.

For the more general case under analysis ϵ is the slope error which arises because the gyroscope rotor spin axis is not vertical. In such an instance a yaw rotation will be accompanied by a gyro-forced roll rotation and a corresponding reduction in pitch. This may readily be understood by considering the extreme case of $\psi_G = 90^o$ and $\theta_G = 0$ as illustrated in Fig. 14. In the launch position the gyroscope rotor spin axis coincides with the z_M axis. After a rotation of 90^o about the z_M axis ($\psi_G = 90^o$) the gyroscope rotor spin axis must still remain fixed in direction relative to the earth and hence the missile will roll by an angle $+ \theta_Q$. At the same time the body pitch will be reduced from $+ \theta_Q$ to zero (missile now flying horizontal) although the gyro pitch angle will not have altered. Any command in pitch given to bring the missile from the launch elevation to the horizontal will therefore, in this case, cause it to dive at the magnitude of the launch elevation.

Using equation (41) θ_T has been plotted as a function of time in Fig. 15 for the special case of $\theta_Q = 20^o$.

θ_E as given by equation (42) (angle of attack of missile not taken into account in the programme fed to the pitch control system) has been plotted in Fig. 16. This graph reveals abrupt discontinuities corresponding to the points B, P and Q of Fig. 8. These discontinuities would not occur in the practical situation and have been introduced because each phase of the trajectory has been assumed to be a steady state one. In practice gradual instead of abrupt changes would occur.

The graph of Fig. 16 shows that substantial errors arise in the "simplified" determination of the Malkara trajectory when a simultaneous yaw manoeuvre is performed. The error term containing the sideslip angle β is zero except when actual rotation in yaw takes place. The remaining error is $\alpha - \epsilon$. Error angle α arises due to neglect of the missile angle of attack in the programme fed to the pitch control system. Error angle ϵ arises because yaw rotation takes place when the gyroscope rotor spin axis is not vertical.

It is clear from Fig. 16 that the slope errors are such as to tend to steer the missile under the target (and hence into the ground) if no manual corrective action is applied. For the particular case analysed the simplified solution for the trajectory slope gives a final slope angle about 8^{o} above that given by the more precise analysis of this paper.

Hence the final slope error angle

$$\theta_E = \alpha - \epsilon = -8^{o}$$

From 8.4.2 the final v.lue of α is -2.3^{o} and hence the final value of ϵ is $+5.7^{o}$.

If the missile angle of attack had been taken into account in the programme fed to the pitch control system such that $\theta_G = \theta_D - \alpha$ then the final slope error would be changed from -8^{o} to -5.7^{o}.

9. CONCLUSION

It has been shown that substantial errors can occur in the estimation of trajectory slope if the method of solution neglects the angle of attack of the missile and the effect of a simultaneous yaw rotation when the gyroscope rotor spin axis is not vertical (which will occur if the missile launch attitude is not horizontal). Moreover the indications of the determination are that a programme based on a "simplified" (Refer to "Definition of Terms") solution to the trajectory would tend to cause the missile to undershoot the target and possibly hit the ground prematurely, if the controller did not make manual correction.

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APPENDIX

Expression for h_o

h_o, the maximum perpendicular distance between the desired path of the missile in pitch and a line drawn through the aimer and the target positions, may be readily derived from the geometric projections drawn in Fig. 9.

h_o = FH (where "FH" represents the distance between points F and H marked in Fig. 8).

$$h_o = FG + GH$$
$$= (FJ - GJ) + (EC - DC)$$

$$FJ = r_1$$

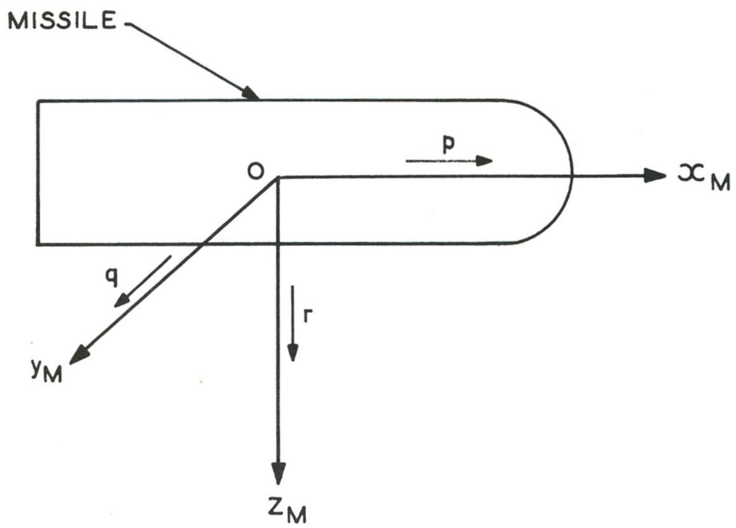
$$GJ = r_1 \cos(\gamma_1 - \gamma_F)$$

$$EC = BC \cos \gamma_F$$
$$= (b \sin \gamma_1 - h) \cos \gamma_F$$

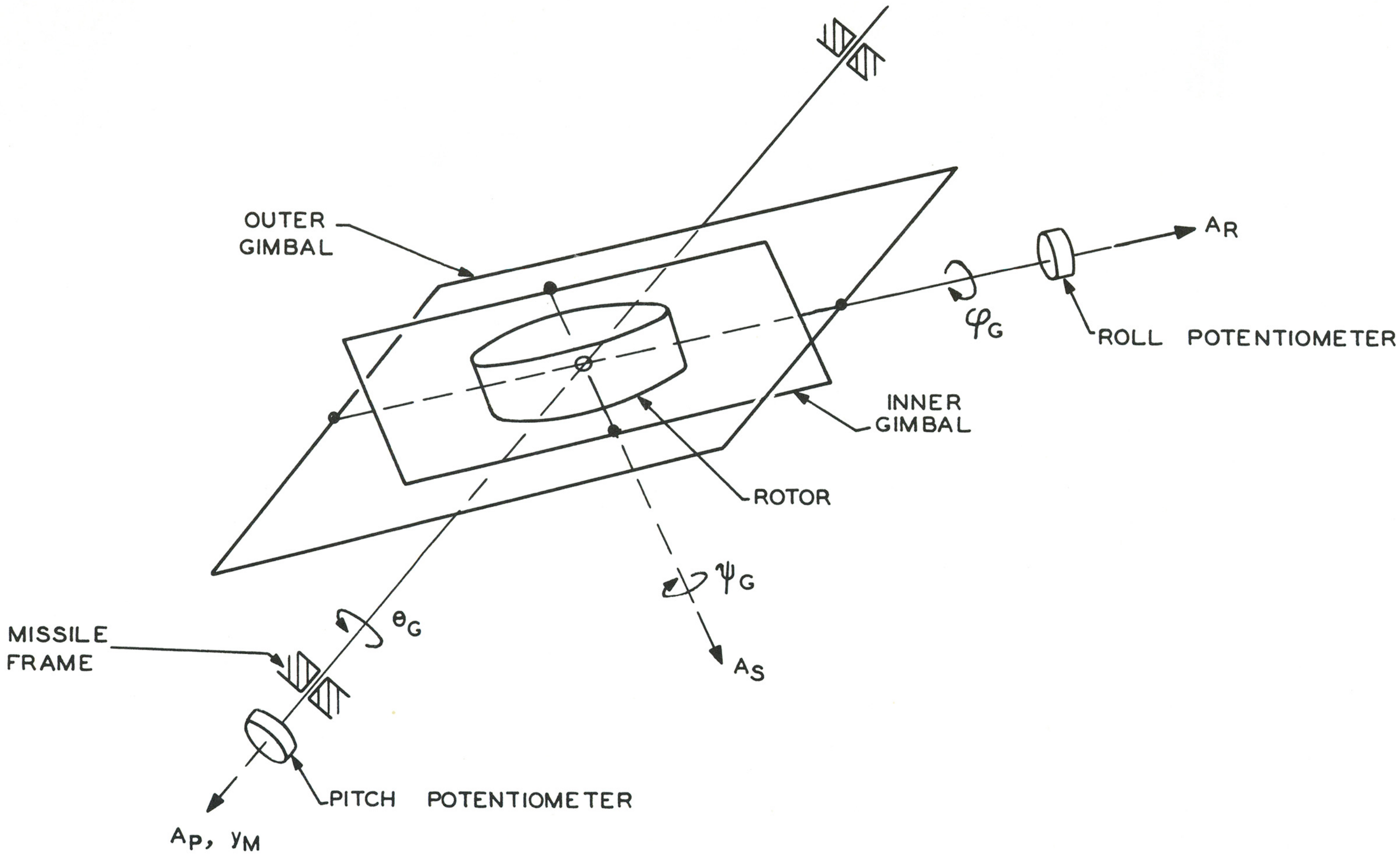
$$DC = AC \sin \gamma_F$$
$$= (b \cos \gamma_1 - a) \sin \gamma_F$$

$$h_o = a \sin\gamma_F - h \cos\gamma_F + b (\sin\gamma_1 \cos\gamma_F - \cos\gamma_1 \sin\gamma_F) + r_1 - r_1 \cos(\gamma_1 - \gamma_F)$$

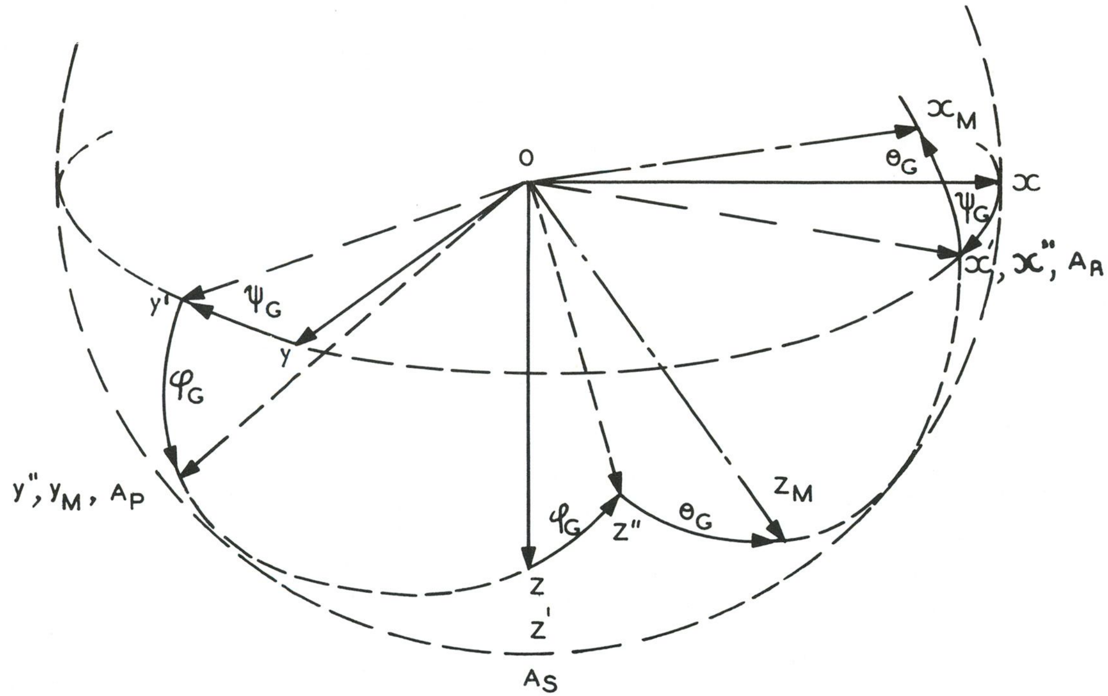
$$h_o = a \sin\gamma_F - h \cos\gamma_F + b \sin(\gamma_1 - \gamma_F) + r_1 \left[1 - \cos(\gamma_1 - \gamma_F)\right]$$



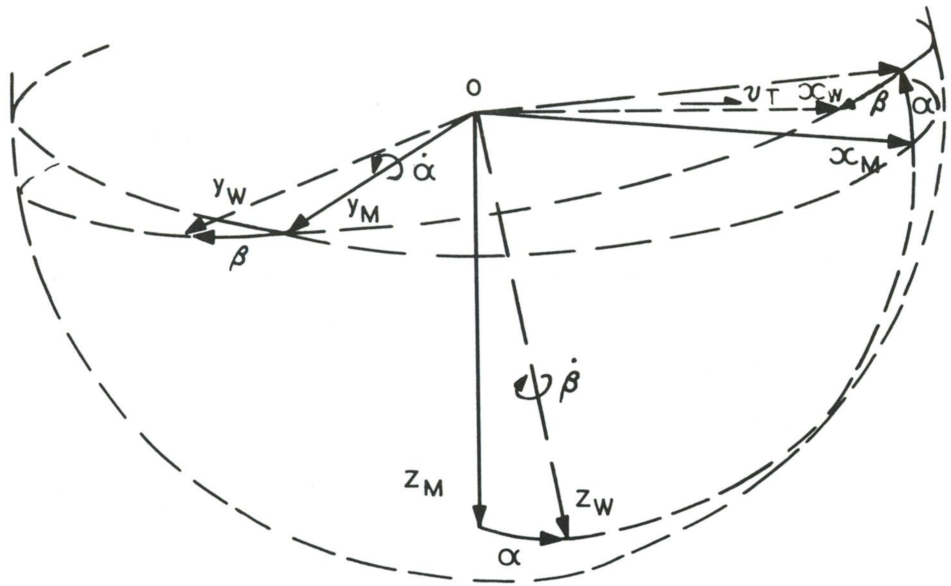
MISSILE AXES



MALKARA GYROSCOPE

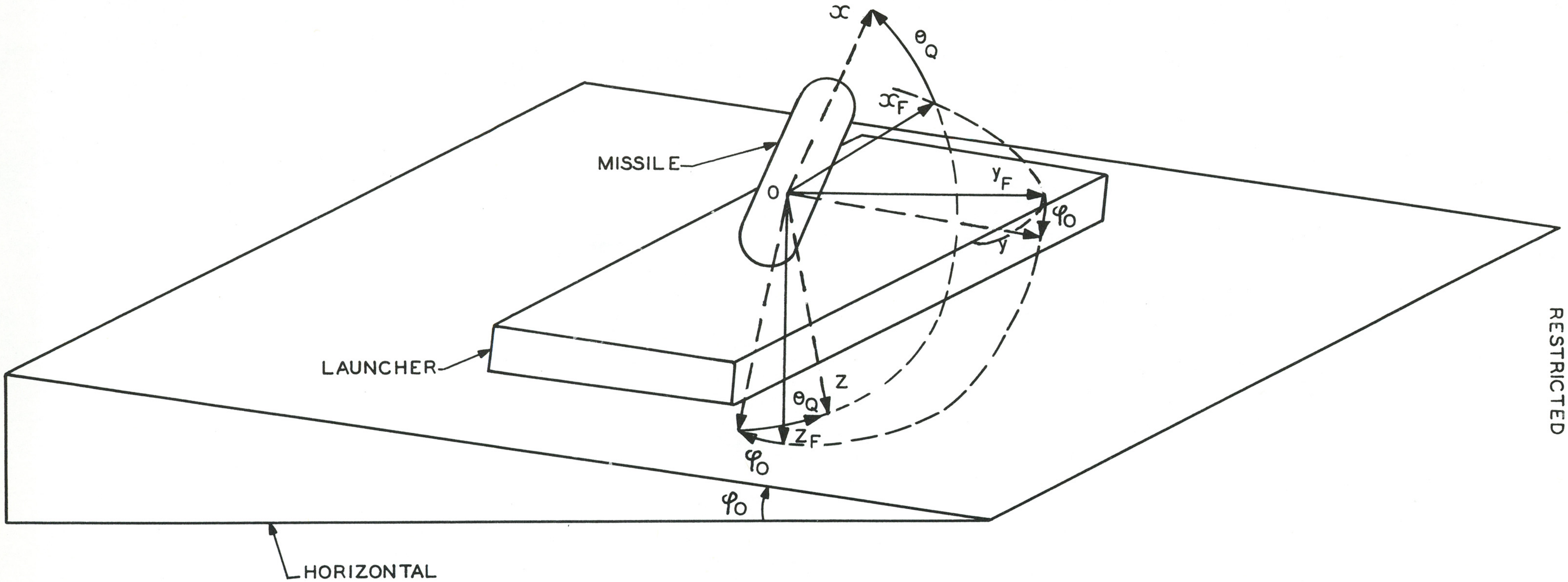


FINITE ROTATIONS OF MISSILE



FLIGHT PATH INCLINATION

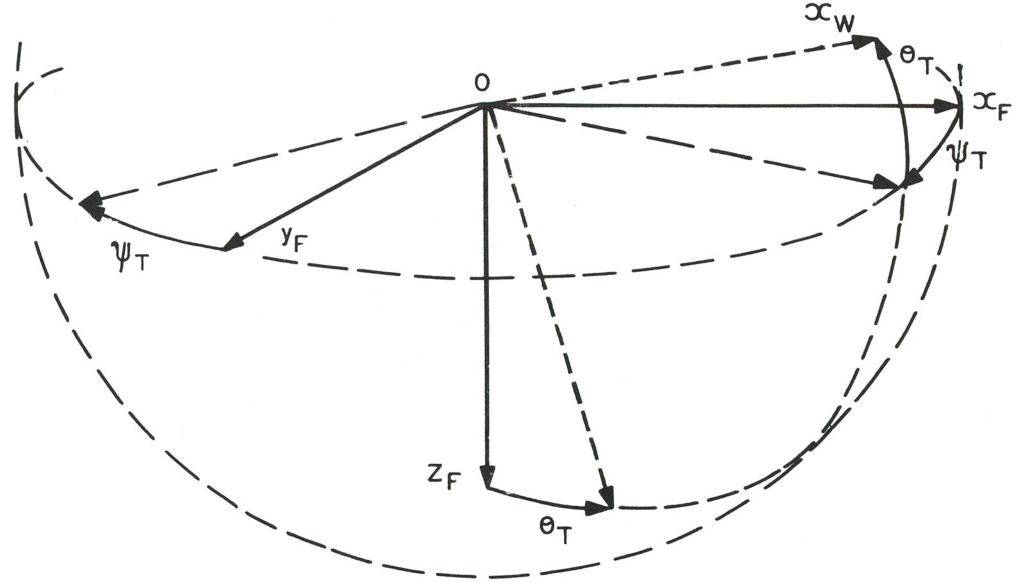
RESTRICTED



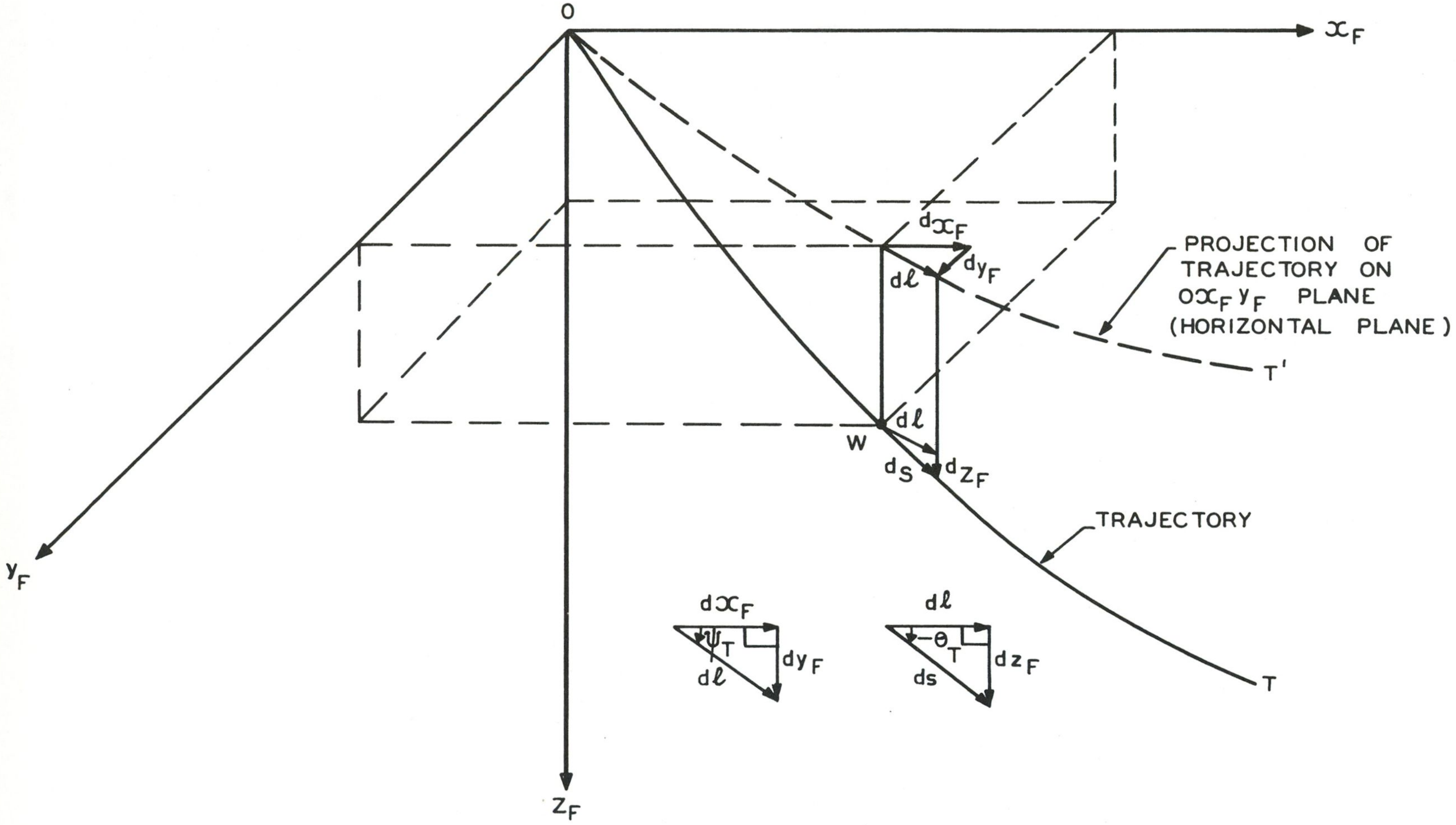
MISSILE LAUNCH ATTITUDE

RESTRICTED

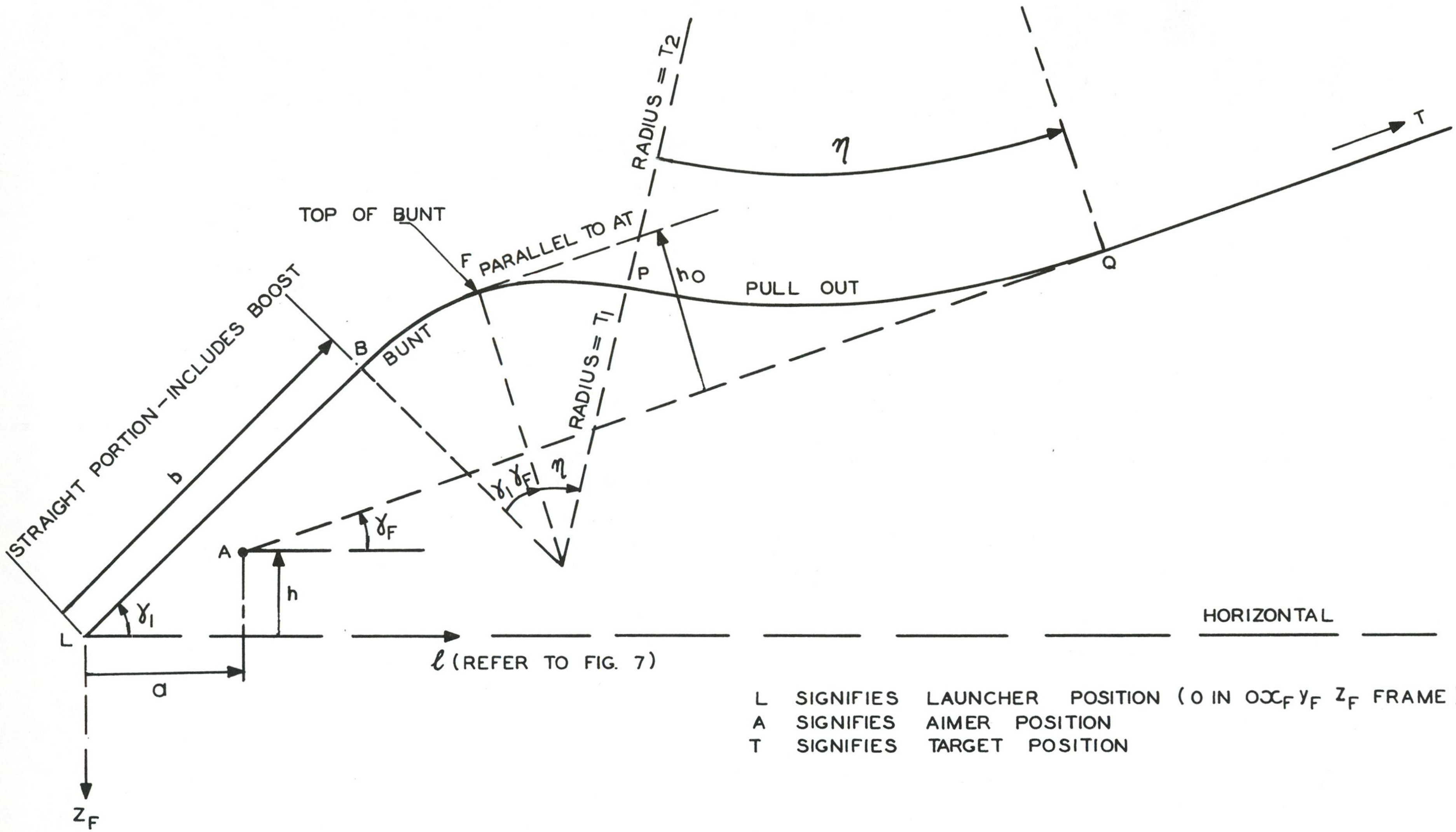
INST. NOTE 69
FIG. 5



TRAJECTORY SPECIFICATION

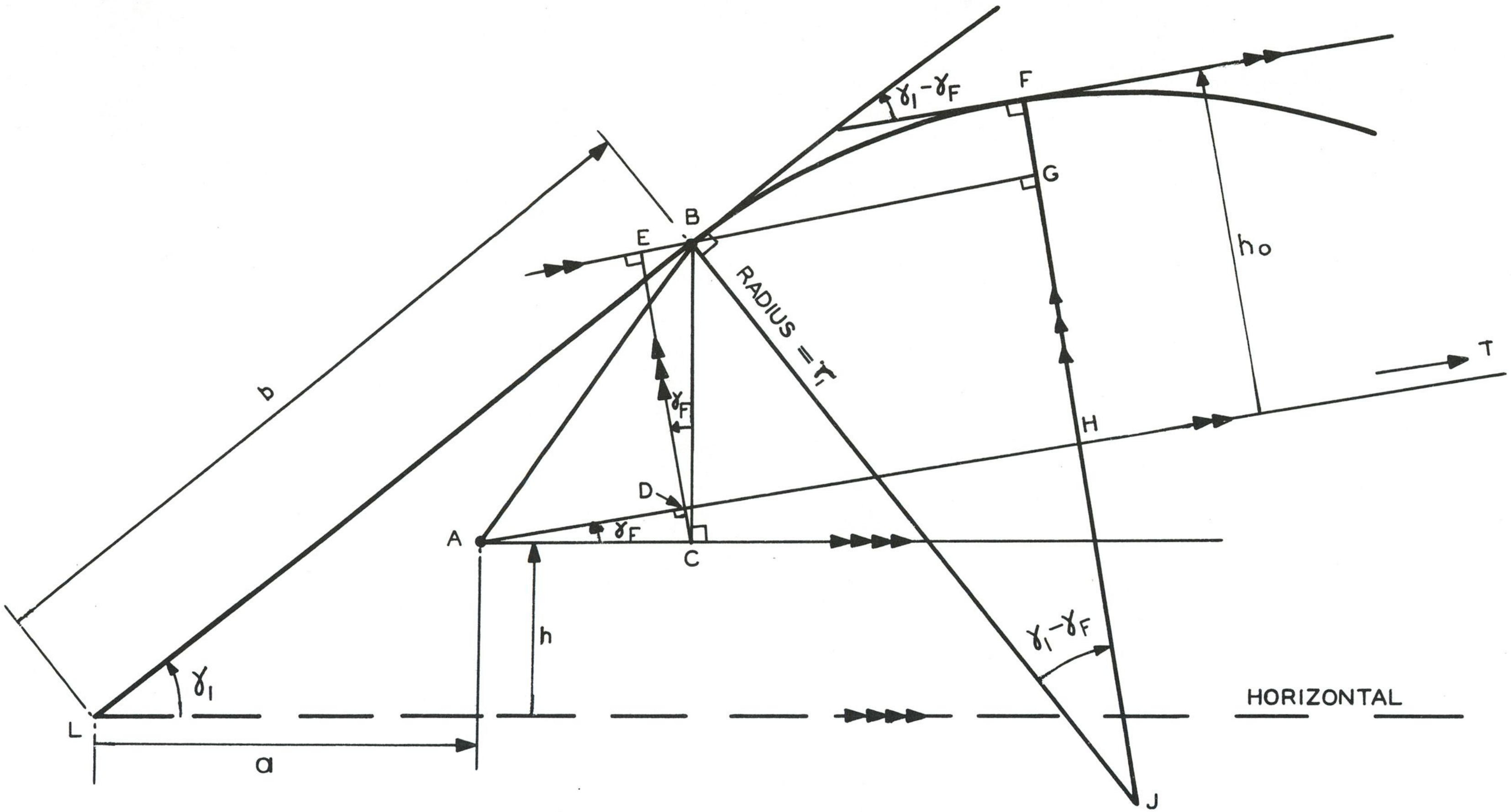


COMPONENTS OF TRAJECTORY DISPLACEMENT IN RELATION TO THE $Ox_Fy_Fz_F$ PLANE



RESTRICTED

DESIRED PATH OF THE MISSILE IN PITCH

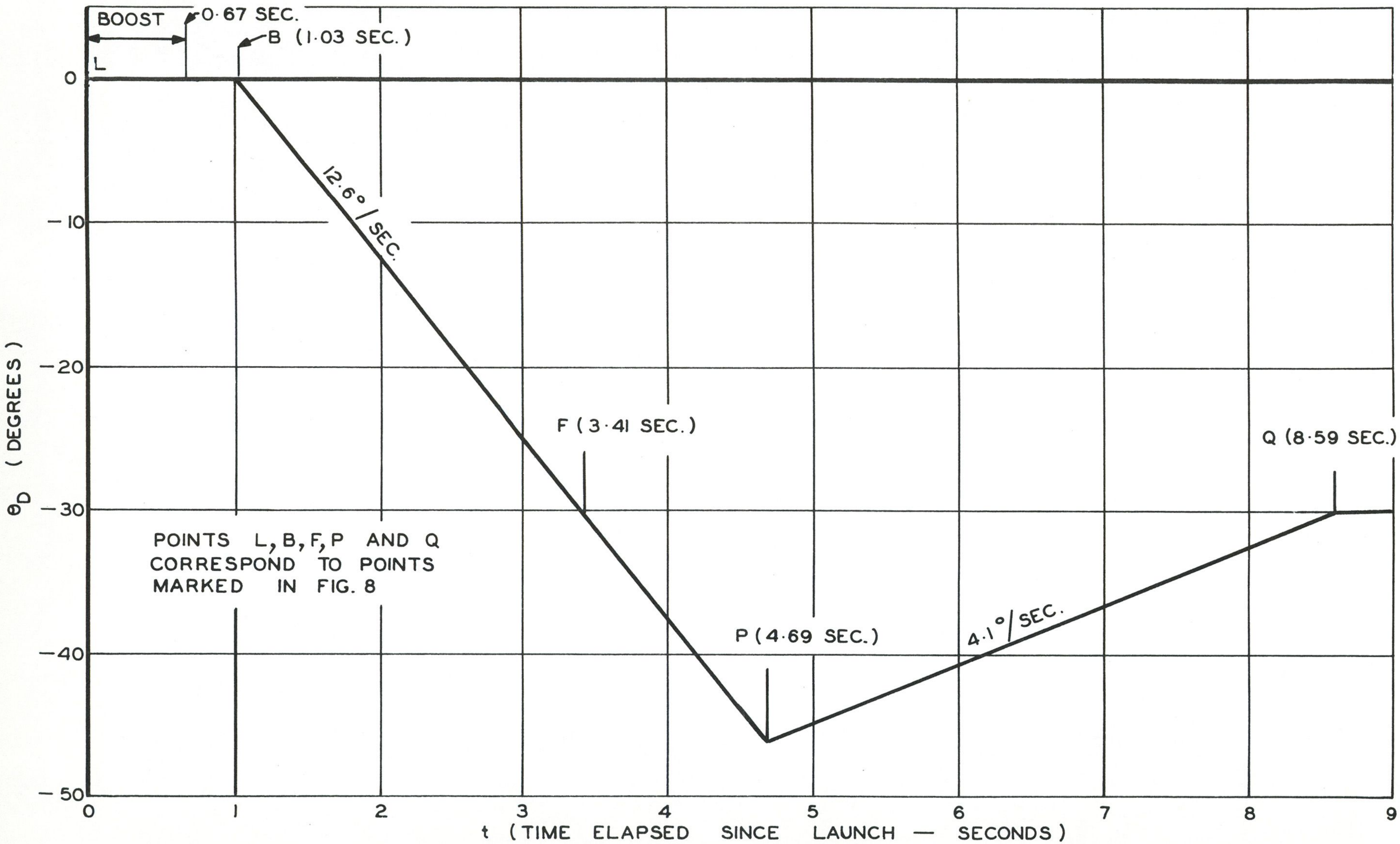


L SIGNIFIES LAUNCHER POSITION
A SIGNIFIES AIMER POSITION
T SIGNIFIES TARGET POSITION

GEOMETRIC PROJECTIONS FOR DETERMINING EXPRESSION FOR h_o

RESTRICTED

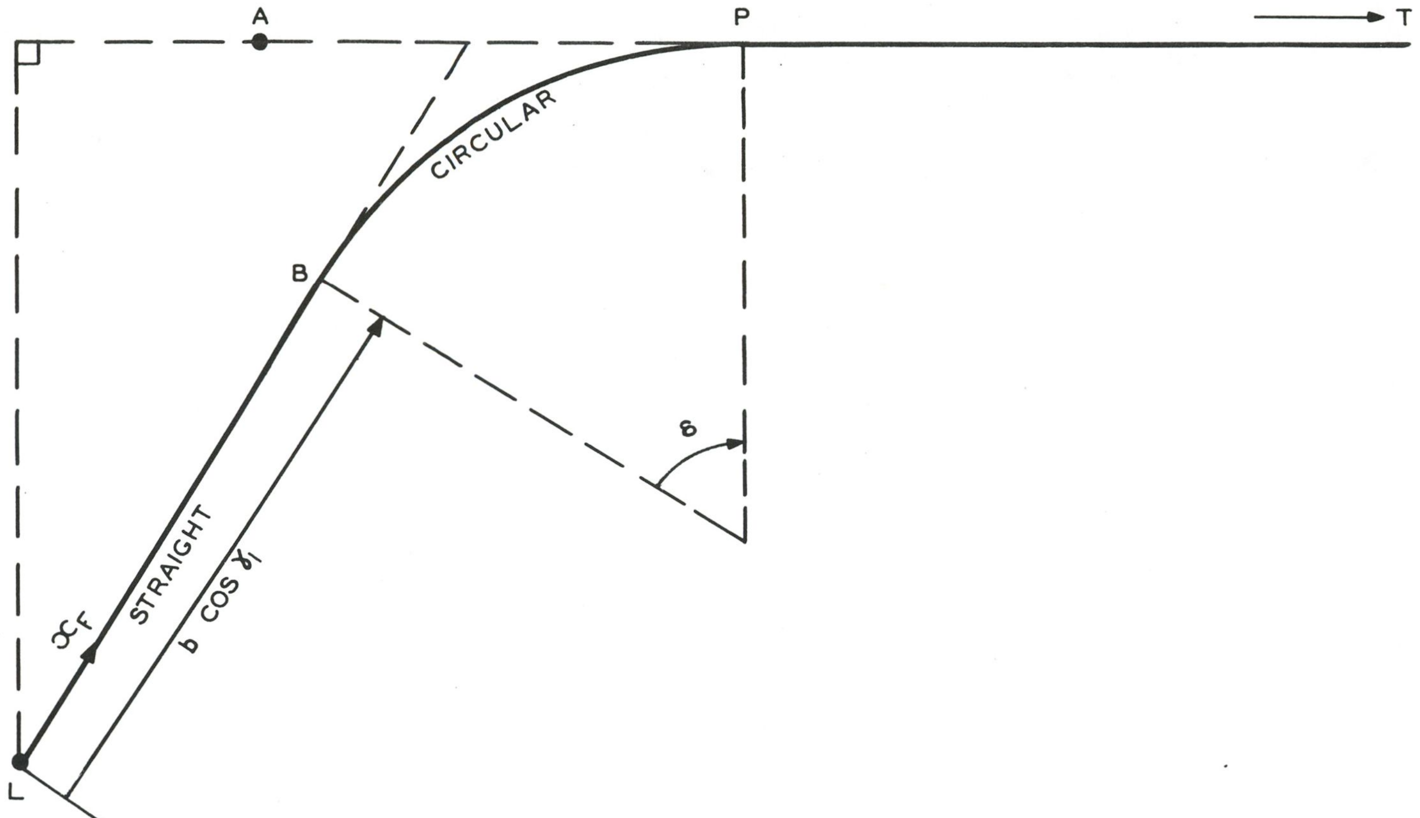
INST. NOTE 69
FIG. 9



PLOT OF θ_D AS A FUNCTION OF TIME

RESTRICTED

INST. NOTE 69
FIG. 10

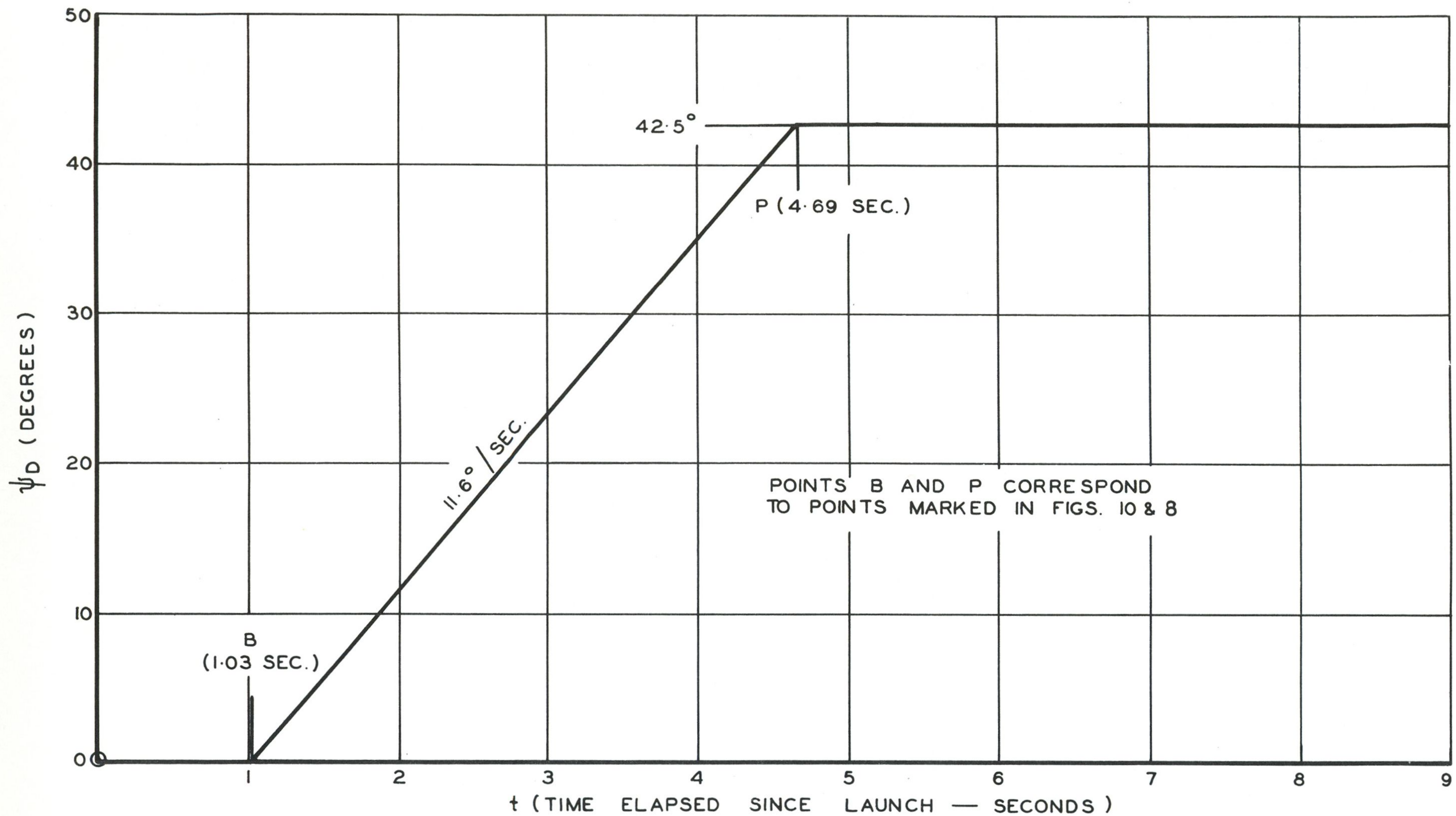


L SIGNIFIES LAUNCHER POSITION (O IN $Ox_F y_F z_F$ FRAME)
A SIGNIFIES AIMER POSITION
T SIGNIFIES TARGET POSITION

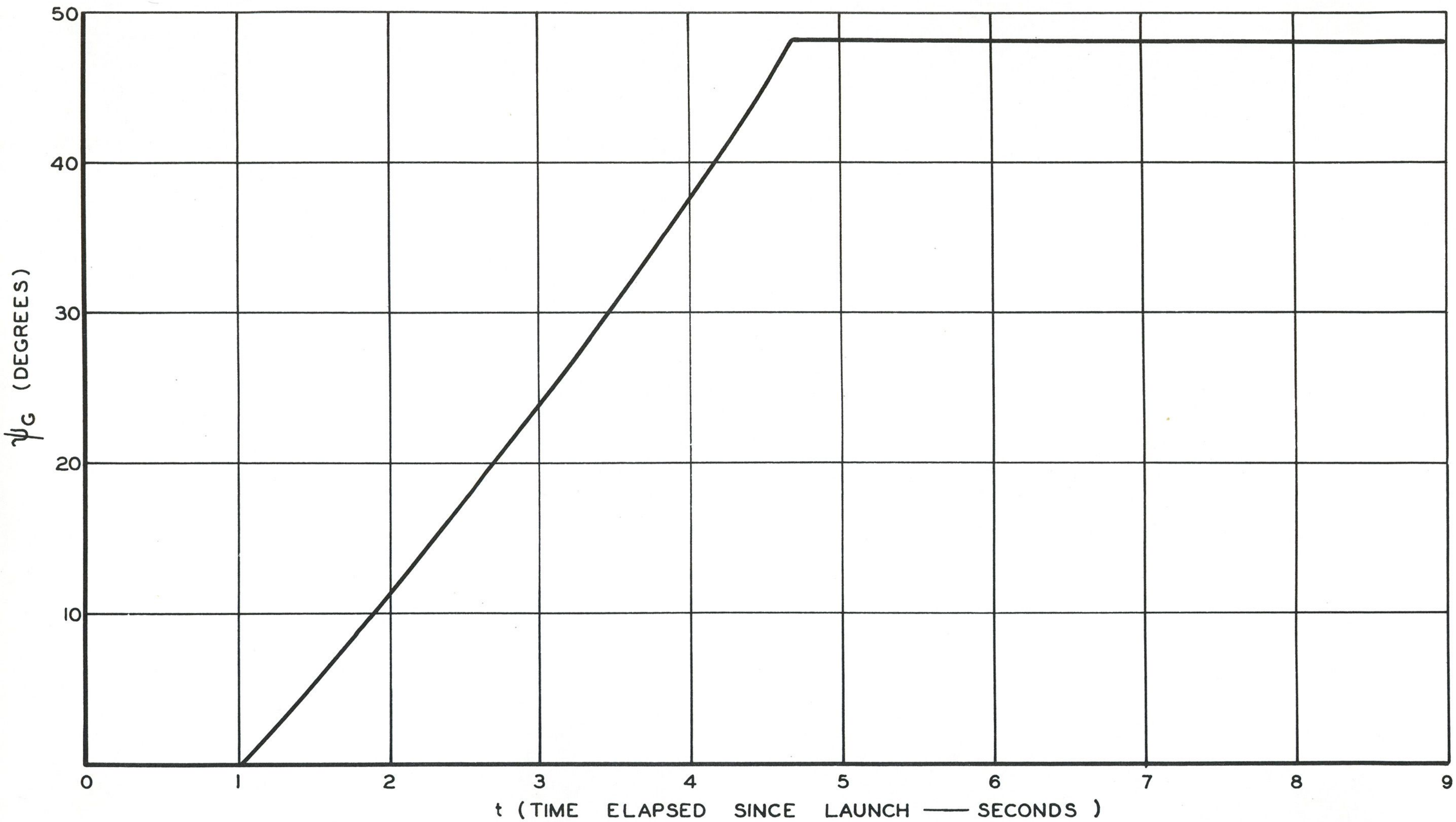
DESIRED PATH OF THE MISSILE IN YAW

RESTRICTED

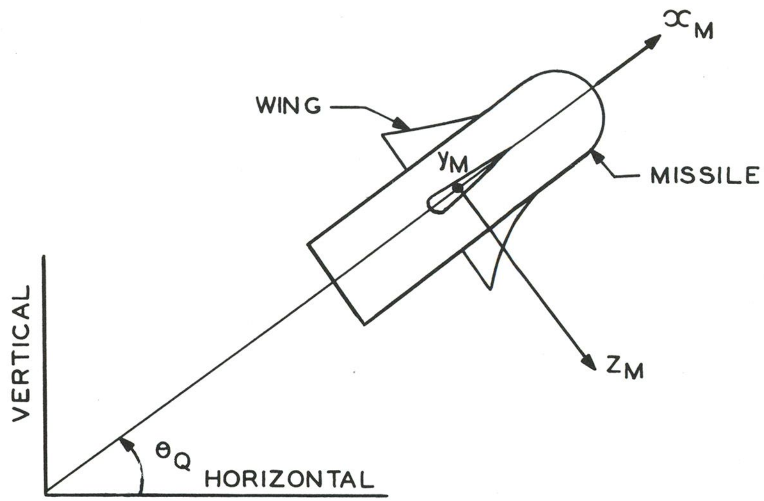
INST. NOTE 69
FIG. 11



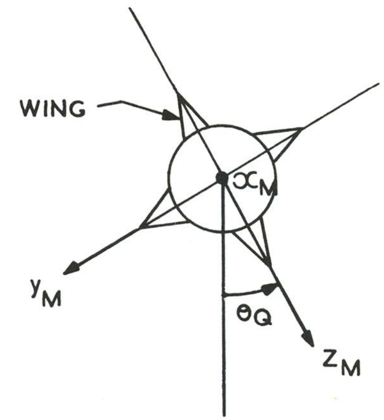
PLOT OF ψ_D AS A FUNCTION OF TIME



PLOT OF ψ_G AS A FUNCTION OF TIME

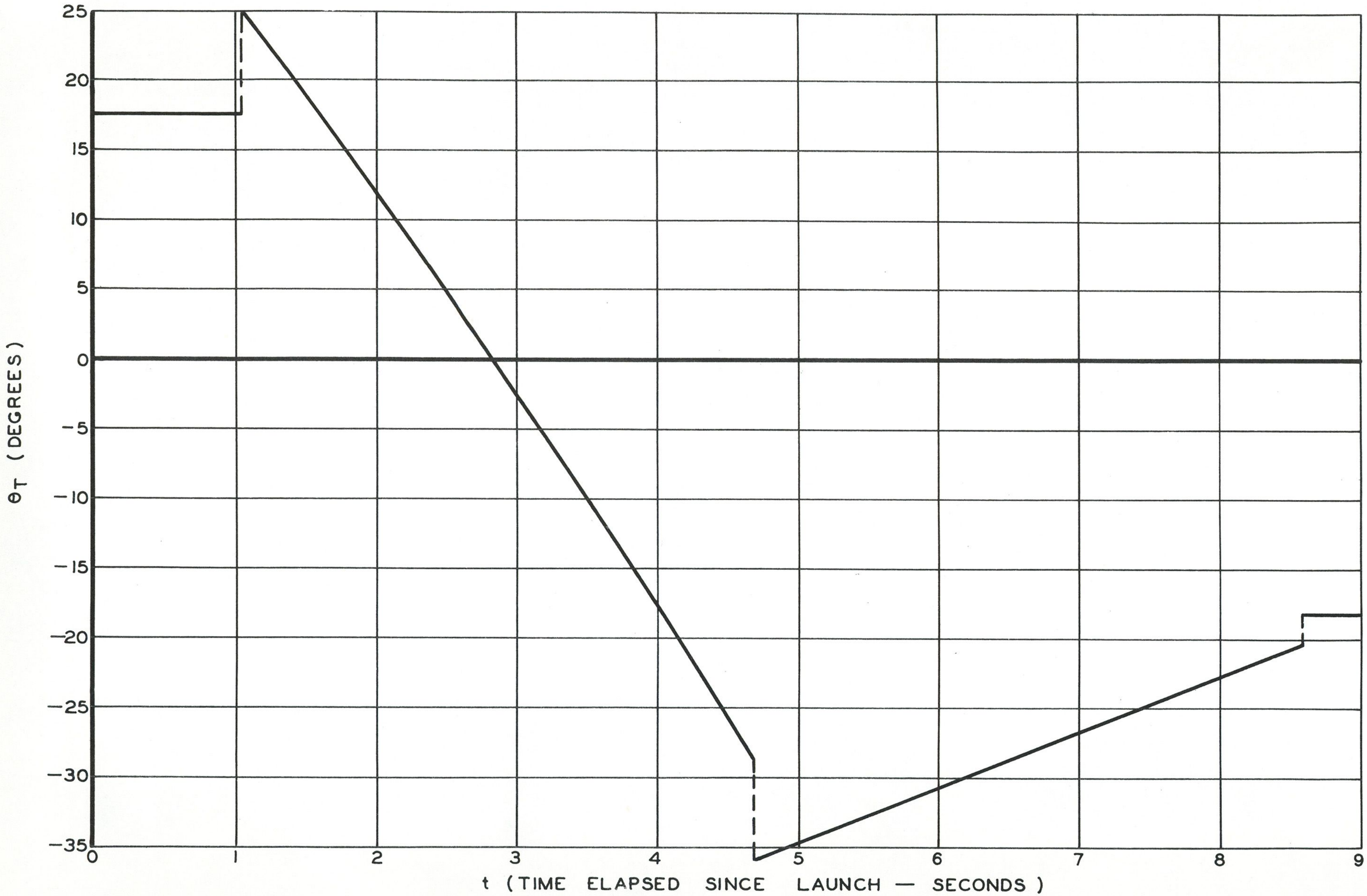


LAUNCH ATTITUDE



ATTITUDE AFTER ROTATION OF 90° ABOUT Z_M

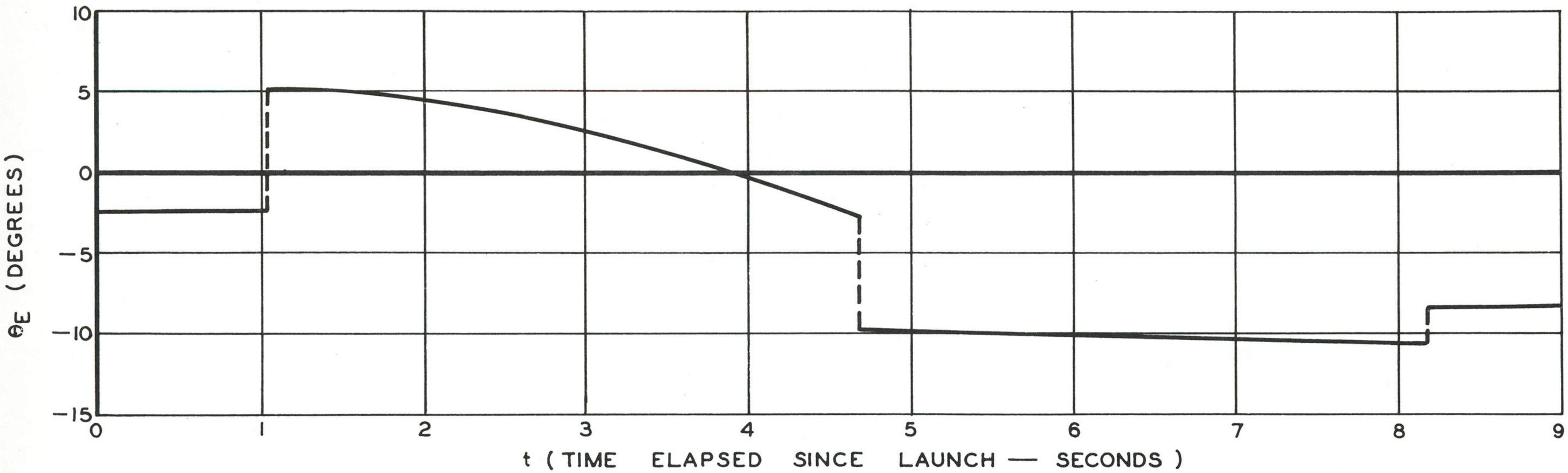
EXTREME CASE ILLUSTRATING INTERCHANGE OF PITCH AND ROLL ROTATIONS



PLOT OF θ_T AS A FUNCTION OF TIME

RESTRICTED

INST. NOTE 69
FIG. 15



RESTRICTED

PLOT OF θ_E AS A FUNCTION OF TIME

INST. NOTE 69
FIG. 16