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RESPONSE OF A BAND PASS FILTER WITH MAXIMALLY FLAT AMPLITUDE CHARACTERISTIC TO A SUDDENLY APPLIED SINUSOIDAL VOLTAGE

by

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SUMMARY

The sharp wavefront of a suddenly applied sinusoidal voltage will not be transmitted through a band pass filter (even if the frequency of the excitation voltage is equal to the centre frequency of the filter) but will experience initial attenuation. It is the purpose of this report to relate the number of cycles which occur during the build up to full amplitude to the bandwidth of the band pass filter. The case of a filter having a maximally flat steady state amplitude response is considered.

CONTENTS

1. INTRODUCTION

The need for filtering abruptly switched sinusoidal voltages arose in the decoding of records obtained using the A.R.L. Flight Memory Equipment. In this application, both cockpit speech and aircraft instrument readings were recorded on a single wire recorder, which had a very restricted bandwidth at the wire speed chosen. Direct recording of speech within a restricted frequency band was employed, while the instrument readings were sampled and recorded as the number of sine wave cycles of a known frequency (outside the speech band) occurring in a burst of these waves.

To extract the instrument readings from the composite recorded signal, it was considered desirable to use a band pass filter to pass the bursts of sine wave oscillations, and at the same time remove the speech. To accurately count the number of recorded cycles occurring during a burst, it was essential that the build-up and decay of the filtered sinusoidal wave trains be as short as possible. (Amplitude variations occurring during playback of the record in the presence of noise make it very difficult to determine the wave train limits with any great accuracy if the build-up and decay times are long.) The speed variations inherent in the recording and replay processes indicated a band pass filter having a pass band broad enough to accommodate the apparent frequency shift due to wow. On the other hand, it was essential that the bandwidth of the filter be made fairly small to ensure adequate filtering of speech, the upper frequency of which was not far removed from the sine wave frequency. As the build-up and decay times of the sinusoidal wave trains are generally increased with decrease in filter bandwidth, a compromise solution had to be reached.

To assist in the specification of a filter, which would decode the instrument readings to a prescribed degree of accuracy, has been the purpose of the following analysis. The relation between the number of cycles of build-up of the filtered sine wave burst, and the bandwidth of the filter, was determined for a practical filter having a maximally flat steady state amplitude response.

2. FOURIER REPRESENTATION OF A SWITCHED SINUSOIDAL VOLTAGE

The frequency spectrum of a sinusoidal voltage which is abruptly switched on and off gives a useful indication of the bandwidth requirements of a band pass filter which is required to pass the voltage signal with reasonable fidelity.

Consider a sinusoidal pulse train defined by $f_1(t) = \sin \omega_0 t$ in the range $-\frac{T}{2} \leqslant t \leqslant \frac{T}{2}$ and being zero elsewhere (refer to Fig. 1).

Let n = number of cycles of the sine wave occurring in time T.

Then $$f_0 = \frac{\omega_0}{2\pi} = \frac{n}{T}$$

Since we are considering a single burst of sine waves (as distinct from a repetitive burst of sine waves) in the present instance, the Fourier Series presentation is not applicable and the Fourier Integral presentation is required (Ref. 1, ch. 20 and Ref. 2, ch. 13).

The Fourier Integral is defined as (Ref. 1):

$$F_1(\omega) = \int_{-\infty}^{\infty} f_1(t)e^{-j\omega t}dt \tag{1}$$

and the corresponding time function is given by

$$f_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\omega)e^{j\omega t}d\omega \tag{2}$$

Some authors (Ref. 2 for instance) place the $\frac{1}{2\pi}$ factor in (1) and exclude it in (2).

The pair of equations (1) and (2) is usually referred to as a "Fourier Transform Pair."

$F_1(\omega)$ may be expressed as

$$F_1(\omega) = a(\omega) - jb(\omega) \tag{3}$$

Expanding equation (1) we obtain:

$$F_1(\omega) = \int_{-\infty}^{\infty} f_1(t)(\cos \omega t - j \sin \omega t)dt$$

For the special case under consideration

$f_1(-t) = -f_1(t)$ and hence $f_1(t)$ is an odd function.

For $f_1(t)$ odd

$$F_1(\omega) = -2j\int_0^\infty f_1(t)\sin\omega t\, dt \tag{4}$$

Substituting $f_1(t) = \sin\omega_0 t$ (up to time $t = T/2$) we obtain

$$F_1(\omega) = -2j\int_0^{T/2} \sin\omega_0 t \sin\omega t\, dt$$

$$= \frac{-2j}{2}\int_0^{T/2} [\cos(\omega_0 - \omega)t - \cos(\omega_0 + \omega)t]dt$$

$$= -j\left|\frac{\sin(\omega_0 - \omega)t}{\omega_0 - \omega} - \frac{\sin(\omega_0 + \omega)t}{\omega_0 + \omega}\right|_0^{T/2}$$

$$= -j\left[\frac{\sin(\omega_0 - \omega)\frac{T}{2}}{\omega_0 - \omega} - \frac{\sin(\omega_0 + \omega)\frac{T}{2}}{\omega_0 + \omega}\right]$$

Comparing with equation (3) it can be seen that

$$a(\omega) = 0$$

$$b(\omega) = \frac{\sin(\omega_0 - \omega)\frac{T}{2}}{\omega_0 - \omega} - \frac{\sin(\omega_0 + \omega)\frac{T}{2}}{\omega_0 + \omega} \tag{5}$$

Expanding equation (2) we can write

$$f_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (-jb(\omega))(\cos\omega t + j\sin\omega t)\, d\omega$$

For $f_1(t)$ real, $Im\, f_1(t)$ must be zero which will be so since $b(\omega)$ is odd.

Hence
$$f_1(t) = \frac{1}{\pi}\int_0^\infty b(\omega)\sin\omega t\, d\omega \tag{6}$$

From (6) it can be seen that $f_1(t)$ may be considered as an infinite sum of infinitesimal components $b(\omega)\Delta(\omega)\sin\omega t$ where $\Delta\omega$ is a small increment in ω. We cannot talk here in terms of discrete frequency components as the contribution to $f_1(t)$ at any particular frequency is infinitesimal. However we can say that the contribution to $f_1(t)$ by components in the band between ω_1 and ω_2 is given by

$$f_1(t) = \frac{1}{\pi}\int_{\omega_1}^{\omega_2} b(\omega)\sin\omega t\, d\omega$$

which is finite.

If in any specific frequency band $b(\omega)$ is relatively high then we can say that this frequency band contributes significantly to $f_1(t)$ whereas if in some other frequency band $b(\omega)$ is relatively small we can say that the contribution to $f_1(t)$ from this band is small. In other words we can say that if we pass $f_1(t)$ through a filter then for reasonable fidelity we require to pass the band of frequencies where $b(\omega)$ is most significant.

Returning to equation (5) and substituting $T = \frac{2\pi n}{\omega_0}$ we obtain:

$$b(\omega) = \frac{1}{\omega_0}\left[\frac{\sin \pi n\left(1 - \frac{\omega}{\omega_0}\right)}{1 - \frac{\omega}{\omega_0}} - \frac{\sin \pi n\left(1 + \frac{\omega}{\omega_0}\right)}{1 + \frac{\omega}{\omega_0}}\right] \tag{7}$$

If it is assumed that n is an integer, or in other words, that an integral number of cycles is considered, then when $\omega = \omega_0$

$$b(\omega_0) = \frac{\pi n}{\omega_0} = \frac{n}{2f_0}$$

since
$$\lim_{\left(1 - \frac{\omega}{\omega_0}\right) \to 0} \frac{\sin \pi n\left(1 - \frac{\omega}{\omega_0}\right)}{1 - \frac{\omega}{\omega_0}} = \pi n$$

For $\omega = \frac{k}{n}\omega_0$ where k is an integer not equal to n, $b(\omega) = 0$ and hence nodes occur.

Consider the particular case for $n = 10$

$$b(\omega) = -\frac{1}{\omega_0}\left[\frac{\sin 10\pi\frac{\omega}{\omega_0}}{1+\frac{\omega}{\omega_0}} + \frac{\sin 10\pi\frac{\omega}{\omega_0}}{1-\frac{\omega}{\omega_0}}\right]$$

If the above equation is normalized by dividing through by $b(\omega_0)$ we obtain

$$\frac{b(\omega)}{b(\omega_0)} = -\frac{\sin 10\pi\frac{\omega}{\omega_0}}{5\pi\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]} \tag{8}$$

In Fig. 2 $\left|\frac{b(\omega)}{b(\omega_0)}\right|$ has been plotted as a function of $\frac{\omega}{\omega_0}$ $\left(\text{which is equal to } \frac{f}{f_0}\right)$ for the frequency range $0 < \frac{\omega}{\omega_0} < 2$.

The band of frequencies associated with the main centre peak of $b(\omega)$ is of interest. The frequency band between the first node on either side of the main peak of $b(\omega)$ will now be calculated.

We have stated that nodes will occur for $\omega = \frac{k}{n}\omega_0$ where k is an integer not equal to n.

The nodes closest to the centre peak wiil occur at $\omega_1 = \frac{n-1}{n}\omega_0$ and $\omega_2 = \frac{n+1}{n}\omega_0$

$$\omega_2 - \omega_1 = \frac{2\omega_0}{n}$$

In other words

$$f_2 - f_1 = \frac{2f_0}{n} \tag{9}$$

For the special case considered here where $n = 10$, $f_2 - f_1 = 0{\cdot}2f_0$.

For a given frequency f_0, the longer the wavetrain the narrower the centre peak becomes. If $f_0 = 3500$ cycles per second and the wavetrain is of 10 cycles duration ($n = 10$), then the width of centre peak (defined as the band of frequencies between the first nodes on either side of f_0) will be 700 cycles per second. In the physical sense, in order to pass this wavetrain with moderate fidelity, we would anticipate a filter bandwidth of at least 700 cycles per second.

3. RESPONSE OF AN IDEALIZED FILTER TO A SWITCHED SINUSOIDAL VOLTAGE

Consider that the sinusoidal pulse train defined above by $f_1(t)$ is passed through a filter which has a voltage transfer ratio given by

$$\frac{e_0}{e_i} = P(\omega) \tag{10}$$

Let $f(t)$ be the time response of the filter and let $F(\omega)$ be the Fourier transform of this time function.

$$F(\omega) = P(\omega)F_1(\omega) \tag{11}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(\omega)F_1(\omega)e^{j\omega t}d\omega \tag{12}$$

Consider an idealized bandpass filter of bandwidth b cycles per second, having unity gain and zero phase shift within the band and infinite attenuation outside the band (Fig. 3). Let f_0 be the centre frequency of the pass band.

$$f(t) = \frac{1}{2\pi}\int_{2\pi(f_0-\frac{1}{2}b)}^{2\pi(f_0+\frac{1}{2}b)} F_1(\omega)e^{j\omega t}d\omega + \frac{1}{2\pi}\int_{-2\pi(f_0+\frac{1}{2}b)}^{-2\pi(f_0-\frac{1}{2}b)} F_1(\omega)e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi(f_0-\frac{1}{2}b)}^{2\pi(f_0+\frac{1}{2}b)} F_1(\omega)(e^{j\omega t} - e^{-j\omega t})d\omega$$

$$\text{since } F_1(-\omega) = -F_1(\omega)$$

$$f(t) = \frac{j}{2\pi}\int_{2\pi(f_0 - \frac{1}{2}b)}^{2\pi(f_0 + \frac{1}{2}b)} F_1(\omega) \sin \omega t \, d\omega \tag{13}$$

$$f(-t) = -f(t).$$

This, as is well known, is not the response of a physical system. It violates a fundamental physical law that a system cannot respond before it is excited. Actually, the above integral may be considered as an approximation to the waveform obtained by considering only a finite portion of the Fourier spectrum, but is of little use in the physical sense. The anomaly arises because the idealized filter assumed above is a physical impossibility. Although in the limit a band pass filter with a maximally flat amplitude response may have an amplitude response similar to the "ideal filter," it can only provide such a response with considerable phase shift throughout the band.

It can be shown that for a transfer function to satisfy the condition of physical realizability, there is a mathematical relationship between the real and imaginary parts of the transfer function, or alternatively, between the amplitude and the phase. For a physical system, it follows that for a given amplitude response, the phase response may be derived, or vice versa. As the derivation of these relationships is rather involved, it will not be included here. For the derivation refer to Ref. 3, p. 48. A summary of the results is given here.

Consider a transfer function $P(\omega)$ defined by

$$P(\omega) = R(\omega) + jS(\omega) = A(\omega)e^{j\phi(\omega)}$$

where $R(\omega)$ is the real part of the transfer function
$S(\omega)$ is the imaginary part of the transfer function
$A(\omega)$ is the amplitude of the transfer function
$\varphi(\omega)$ is the phase of the transfer function

$$\text{then } R(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{S(u)}{u-\omega} du \tag{14}$$

$$\text{and } S(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{R(u)}{u-\omega} du \tag{15}$$

$$ln\, A(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\varphi(u)}{u-\omega} du \tag{16}$$

$$\text{and } \varphi(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{ln\, A(u)}{u-\omega} du \tag{17}$$

Note that $R(\omega)$ and $A(\omega)$ are even functions whereas $S(\omega)$ and $\varphi(\omega)$ are odd functions. Expressed analytically

$$R(-\omega) = R(\omega)$$
$$S(-\omega) = -S(\omega)$$
$$A(-\omega) = A(\omega)$$
$$\text{and } \varphi(-\omega) = -\varphi(\omega)$$

4. REALIZABLE FILTER REPRESENTATION

A filter having a maximally flat amplitude response within the pass band will be considered. To obtain the transfer function of such a filter a low pass filter with a maximally flat amplitude response will first be considered and then a suitable low pass to band pass transformation will be applied.

4.1. Low Pass Filter with Maximally Flat Amplitude Response

Assume a low pass filter has a transfer function

$$P_1(S) = \frac{K}{a_0 + a_1S + a_2S^2 + \ldots + S^n}$$

It is convenient to normalize the above function with respect to its D.C. value.

$$\text{Define } P(S) = \frac{P_1(S)}{P_1(O)} = \frac{1}{1 + \left(\frac{a_1}{a_0}\right)S + \left(\frac{a_2}{a_0}\right)S^2 + \ldots + \frac{1}{a_0}S^n}$$

$$|P(j\omega)| = [P(j\omega)P(-j\omega)]^{\frac{1}{2}} \tag{18}$$

which takes the form

$$|P(j\omega)| = \frac{1}{(1 + C_1\omega^2 + C_2\omega^4 + \ldots + C_n\omega^{2n})^{\frac{1}{2}}} \tag{19}$$

To obtain a maximally flat amplitude response (often referred to as a Butterworth response) at $\omega = 0$ we require as many as possible partial derivatives of $|P|$ (abbreviation for $|P(i\omega)|$) with respect to ω $\left(\text{i.e. } \frac{\partial |P|}{\partial\omega}, \frac{\partial^2|P|}{\partial\omega^2}, \frac{\partial^3|P|}{\partial\omega^3} \ldots\right)$ to vanish at $\omega = 0$ (Ref. 4, p. 152).

If we define

$$Q = 1 + C_1\omega^2 + C_2\omega^4 + \ldots + C_n\omega^{2n} \tag{20}$$

then equation (19) may be rewritten as

$$|P(j\omega)| = |P| = Q^{-\frac{1}{2}}$$

Taking partial derivatives of $|P|$ with respect to ω we obtain

$$\frac{\partial|P|}{\partial\omega} = \frac{\partial|P|}{\partial Q}\frac{\partial Q}{\partial\omega}$$

$$\frac{\partial^2|P|}{\partial\omega^2} = \frac{\partial|P|}{\partial Q}\frac{\partial^2 Q}{\partial\omega^2} + \left(\frac{\partial Q}{\partial\omega}\right)^2\frac{\partial^2|P|}{\partial Q^2}$$

$$\frac{\partial^3|P|}{\partial\omega^3} = \frac{\partial|P|}{\partial Q}\frac{\partial^3 Q}{\partial\omega^3} + 3\frac{\partial Q}{\partial\omega}\cdot\frac{\partial^2 Q}{\partial\omega^2}\cdot\frac{\partial^2|P|}{\partial Q^2} + \left(\frac{\partial Q}{\partial\omega}\right)^3\frac{\partial^3|P|}{\partial Q^3}$$

$$\frac{\partial^N|P|}{\partial\omega^N} = \frac{\partial|P|}{\partial Q}\frac{\partial^N Q}{\partial\omega^N} + \text{Terms containing } \frac{\partial^k Q}{\partial\omega^k} \text{ as factors where } 1 \leqslant k \leqslant N-1.$$

Hence all partial derivatives of $|P|$ with respect to ω, up to the kth, will be zero at $\omega = 0$ if all partial derivatives of Q with respect to ω, up to the kth, are zero at $\omega = 0$.

All odd order derivatives of Q with respect to ω $\left(\frac{\partial Q}{\partial\omega}, \frac{\partial^3 Q}{\partial\omega^3} \ldots\right)$ will be zero at $\omega = 0$ irrespective of the values of the coefficients C_1, C_2 etc. (check by differentiation of Q).

Differentiation of equation (20) yields the following derivatives:

$$\left.\frac{\partial^2 Q}{\partial\omega^2}\right|_{\omega=0} = 2C_1$$

$$\left.\frac{\partial^4 Q}{\partial\omega^4}\right|_{\omega=0} = 2.3.4.C_2$$

$$\left.\frac{\partial^{2k} Q}{\partial\omega^{2k}}\right|_{\omega=0} = 2.3.4 \ldots 2kC_k$$

If all coefficients C_1, C_2, etc., up to but excluding the nth are made equal to zero then

$$\left.\frac{\partial^k Q}{\partial\omega^k}\right|_{\omega=0} = 0$$

and hence also $\left.\frac{\partial^k|P|}{\partial\omega^k}\right|_{\omega=0} = 0$

for $1 \leqslant k \leqslant 2(n-1)$ (considering even order derivatives only).

In such a case $|P|$ is said to have $(n-1)$th order maximal flatness at $\omega = 0$ (see also Ref. 4, p. 154).

If $|P|$ is made $(n-1)$th order maximally flat then equation (10) simplifies to

$$|P(j\omega)| = \frac{1}{(1 + C_n\omega^{2n})^{\frac{1}{2}}} \tag{21}$$

The low pass function $|P(j\omega)|$ defined by equation (21) may be referred to as a Butterworth function of order n (Ref. 5, p. 427), but it must be realized that it has $(n-1)$th order maximal flatness.

Putting $C_n = \frac{1}{b^{2n}}$

$$|P(j\omega)| = \frac{1}{\left[1 + \left(\frac{\omega}{b}\right)^{2n}\right]^{\frac{1}{2}}} \tag{22}$$

Putting $S = j\omega$ in equation (22) and using equation (18) we obtain:

$$P(S)P(-S) = \frac{1}{1 + (-1)^n\left(\frac{S}{b}\right)^{2n}} \tag{23}$$

The poles of the real function $P(S)$ lie in the L.H. half plane whilst those of $P(-S)$ lie in the R.H. half plane. It is convenient to consider the poles of $P(S)P(-S)$ and then take only those which lie in the L.H. half plane to obtain the real function $P(S)$.

To determine the poles of $P(S)$ we have to solve the equation

$$1 + (-1)^n\left(\frac{S}{b}\right)^{2n} = 0$$

$$\left(\frac{S}{b}\right)^{2n} = (-1)^{n+1}$$

For n even

$$\left(\frac{S}{b}\right)^{2n} = \text{cis } \pi(2N + 1) \text{ where } N \text{ is an integer}$$

$$S = b \text{ cis } \frac{\pi}{2n}(2N + 1) \tag{24}$$

For n odd

$$\left(\frac{S}{b}\right)^{2n} = \text{cis } 2N\pi$$

$$S = b \text{ cis } \frac{\pi N}{n} \tag{25}$$

The poles of the function $P(S)P(-S)$ have been plotted in Fig. 4 for the special cases $n = 5$ and $n = 6$.

4.2. Bandwidth and Attenuation Slopes of Low Pass Filter

The $-3db$ bandwith limits occur when

$$\frac{|P(j\omega)|}{|P(j\omega)|_{\omega=0}} = \frac{1}{\sqrt{2}}$$

$$\frac{1}{\left[1 + \left(\frac{\omega}{b}\right)^{2n}\right]^{\frac{1}{2}}} = \frac{1}{\sqrt{2}}$$

The above equation has the solution $\omega = b$ and hence the bandwidth of the filter is b rad/sec.

If we designate the amplitude by A, or in other words $A = |P(j\omega)|$, then the attenuation slope at the half power point is given by

$$\left.\frac{dA}{d\omega}\right|_{\omega=b} = -\left.\frac{n\omega^{2n-1}}{b^{2n}\left[1 + \left(\frac{\omega}{b}\right)^{2n}\right]^{3/2}}\right|_{\omega=b}$$

$$= -\frac{n}{2\sqrt{2}b} \tag{26}$$

The response curve is shown diagramatically in Fig. 5.

It is desirable to express the slope in logarithmic units of db/octave.

$$\text{Put } Y = 20 \log_{10} A \quad = 20 \log_{10} e \ln A$$

$$X = \log_2 \omega \quad = \log_2 e \ln \omega$$

$$\frac{dY}{dX} = \frac{dY}{dA}\frac{dA}{d\omega}\frac{d\omega}{dX}$$

$$\frac{dY}{dA} = \frac{20 \log_{10} e}{A}$$

$$\frac{dX}{d\omega} = \frac{\log_2 e}{\omega}$$

$$\frac{dY}{dX} = \frac{20 \log_{10} e}{\log_2 e}\frac{dA}{d\omega}\frac{\omega}{A}$$

$$= 20 \log_{10} 2 \frac{dA}{d\omega}\frac{\omega}{A}$$

$$\frac{dY}{dX} \simeq 6\frac{\omega}{A}\frac{dA}{d\omega} \; db/\text{octave} \tag{27}$$

For graphical representation refer to Fig. 6.

For nth order Butterworth low pass filter at $\omega = b$

$$\frac{dY}{dX} = -\frac{6bn}{\frac{1}{\sqrt{2}}2\sqrt{2}b} = -3n \; db/\text{octave} \tag{28}$$

4.3. Low Pass to Band Pass Transformation

A justification for the complex plane transformation to be employed may be seen with the aid of a simple circuit.

The admittance of a parallel GLC circuit is

$$Y(S) = SC + \frac{1}{SL} + G$$

$$= C\omega_0\left(\frac{S}{\omega_0} + \frac{\omega_0}{S}\right) + G \tag{29}$$

$$\text{where } \omega_0 = \frac{1}{\sqrt{LC}}$$

If a transformation of the form

$$Z = \omega_0\left(\frac{S}{\omega_0} + \frac{\omega_0}{S}\right) \tag{30}$$

is used then equation (29) becomes

$$Y = CZ + G$$

which is recognized as the admittance of a shunt RC circuit in a new frequency domain. Note that the shunt GLC circuit is a simple band pass circuit and the shunt RC circuit is a simple low pass circuit.

The above transformation may be applied to the transfer function of the maximally flat low pass filter (Ref. 5, p. 428 and Ref. 6). For the low pass filter we may write using equation (23):

$$P(Z)P(-Z) = \frac{1}{1 + (-1)^n\left(\frac{Z}{b}\right)^{2n}} \tag{31}$$

If we let $B(S)$ be the transfer function of the new band pass filter then

$$B(S)B(-S) = \frac{1}{1 + (-1)^n\left[\frac{\omega_0}{b}\left(\frac{S}{\omega_0} + \frac{\omega_0}{S}\right)\right]^{2n}} \tag{32}$$

Putting $Z = u + jv$

$S = x + jy$

$$u + jv = \frac{x(x^2 + y^2 + \omega_0^2) + jy(x^2 + y^2 - \omega_0^2)}{x^2 + y^2}$$

$$u = \frac{x(x^2 + y^2 + \omega_0^2)}{x^2 + y^2} \tag{33}$$

$$v = \frac{y(x^2 + y^2 - \omega_0^2)}{x^2 + y^2} \tag{34}$$

For $y = 0$, $v = 0$. Hence the imaginary axis of the Z plane transforms into the imaginary axis of the S plane.

For $x \leqslant 0$, $u \leqslant 0$, and hence the left hand half plane of Z transforms into the left hand half plane of S.

Now from equation (30) $S = \frac{Z}{2} \pm \sqrt{\left(\frac{Z}{2}\right)^2 - \omega_0^2}$

Each point on the Z plane transforms into two points on the S plane.

For $Z = jv$

$$S = j\left[\frac{v}{2} \pm \sqrt{\left(\frac{v}{2}\right)^2 + \omega_0^2}\right] = j\omega$$

$$\omega = \frac{v}{2} \pm \sqrt{\omega_0^2 + \left(\frac{v}{2}\right)^2}$$

The correspondence between v and ω can be seen from the following table where the two solutions to ω at $v = v_C$ are designated by ω_A and $-\omega_B$ respectively.

v	ω
0	$\pm\,\omega_0$
v_C	$\omega_A = \frac{v_C}{2} + \sqrt{\omega_0^2 + \left(\frac{v_C}{2}\right)^2}$ $-\omega_B = \frac{v_C}{2} - \sqrt{\omega_0^2 - \left(\frac{v_C}{2}\right)^2}$
$-\,v_C$	$\omega_B = -\frac{v_C}{2} + \sqrt{\omega_0^2 + \left(\frac{v_C}{2}\right)^2}$ $-\omega_A = -\frac{v_C}{2} - \sqrt{\omega_0^2 + \left(\frac{v_C}{2}\right)^2}$

Hence from the above table:

$$\omega_A - \omega_B = v_C$$
$$\omega_A \omega_B = \omega_0^2$$

4.4. Bandwidth and Slope of Band Pass Filter

Putting $S = j\omega$ in equation (32) and using equation (18) we obtain:

$$|B(j\omega| = \frac{1}{\sqrt{\left[1 + (-1)^n\left\{\frac{j\omega_0}{b}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right\}^{2n}\right]}}$$

$$= \frac{1}{\sqrt{\left[1 + \left\{\frac{\omega_0}{b}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right\}^{2n}\right]}} \qquad (35)$$

$|B(j\omega)|_{max.}$ occurs when $\omega = \omega_0$ and is equal to 1. f_0 (equal to $\omega_0/2\pi$) will be referred to as the "centre frequency" of the band pass filter.

The $3db$ bandwidth may be determined from the relationship

$$|B(j\omega)| = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{\left(1 + \left[\frac{\omega_0}{b}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]^{2n}\right)}}$$

$$\frac{\omega_0}{b}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = \pm 1$$

$$\omega = \frac{+b \pm \sqrt{b^2 + 4\omega_0^2}}{2} \quad \text{or} \quad \frac{-b \pm \sqrt{b^2 + 4\omega_0^2}}{2}$$

Let f_2 be the upper half power frequency and let f_1 be the lower half power frequency. Neglecting negative solutions for ω we obtain:

then
$$\omega_2 = 2\pi f_2 = \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + \omega_0^2} \qquad (36)$$

$$\omega_1 = 2\pi f_1 = -\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + \omega_0^2} \qquad (37)$$

$$\omega_2 - \omega_1 = b$$

The bandwidth of the band pass filter is equal to the bandwidth of the corresponding low pass filter.

The slope of the band pass filter amplitude characteristic at the half power points is given by

$$\left.\frac{d|B(j\omega)|}{d\omega}\right|_{\omega=\omega_1} = \frac{n}{2\sqrt{2}b}\left[1+\left(\frac{\omega_0}{\omega_1}\right)^2\right] \quad (38)$$

$$\left.\frac{d|B(j\omega)|}{d\omega}\right|_{\omega=\omega_2} = -\frac{n}{2\sqrt{2}b}\left[1+\left(\frac{\omega_0}{\omega_2}\right)^2\right] \quad (39)$$

In Fig. 7 the frequency response of a maximally flat band pass filter as given in equation (35) has been plotted for the particular case of $b/\omega_0 = 0{\cdot}2$. The range $1 \leqslant n \leqslant 4$ has been taken.

4.5. Check for Maximal Flatness of Band Pass Filter

To check whether the transformation used in Sec. 4.3 gives a band pass filter which has a maximally flat amplitude characteristic at $\omega = \omega_0$ it is necessary to evaluate the partial derivatives of $|B(j\omega)|$ (or $|B|$) with respect to ω at $\omega = \omega_0$.

The band pass filter is said to have $(n-1)$th order maximal flatness at $\omega = \omega_0$ if

$$\left.\frac{\partial^k |B|}{\partial \omega^k}\right|_{\omega=\omega_0} = 0 \qquad \text{for } 1 \leqslant k \leqslant 2(n-1).$$

Putting $Z = jv$ in equation (31) and using equation (18) we obtain:

$$|B(j\omega)| = |B| = \frac{1}{\left[1+\left(\frac{v}{b}\right)^{2n}\right]^{\frac{1}{2}}} \quad (40)$$

If $|P|$ is substituted for $|B|$ and ω is substituted for v in equation (40) we obtain the low pass function of equation (21) which has been checked for maximal flatness in Sec. 4.1.

Taking partial derivatives of $|B|$ with respect to ω we obtain the following:

$$\frac{\partial |B|}{\partial \omega} = \frac{\partial |B|}{\partial v}\,\frac{\partial v}{\partial \omega}$$

$$\frac{\partial^2 |B|}{\partial \omega^2} = \frac{\partial^2 |B|}{\partial v^2}\left(\frac{\partial v}{\partial \omega}\right)^2 + \frac{\partial |B|}{\partial v}\,\frac{\partial^2 v}{\partial \omega^2}$$

$$\frac{\partial^3 |B|}{\partial \omega^3} = \frac{\partial^3 |B|}{\partial v^3}\left(\frac{\partial v}{\partial \omega}\right)^3 + \frac{\partial |B|}{\partial v}\,\frac{\partial^3 v}{\partial \omega^3} + \frac{3\partial^2 |B|}{\partial v^2}\,\frac{\partial v}{\partial \omega}\,\frac{\partial^2 v}{\partial \omega^2}$$

$$\frac{\partial^N |B|}{\partial \omega^N} = \frac{\partial^N |B|}{\partial v^N}\left(\frac{\partial v}{\partial \omega}\right)^N + \text{Terms containing } \frac{\partial^k |B|}{\partial v^k} \text{ as factors where } 1 \leqslant k \leqslant N-1.$$

Substituting $Z = jv$ and $S = j\omega$ in equation (30) we obtain

$$v = \omega_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) \quad (41)$$

$$\frac{\partial v}{\partial \omega} = 1 + \left(\frac{\omega_0}{\omega}\right)^2$$

$$\left.\frac{\partial v}{\partial \omega}\right|_{\omega=\omega_0} = 2 \quad \text{(i.e. non-zero).}$$

From Sec. 4.1 we know that

$$\left.\frac{\partial^k |B|}{\partial v^k}\right|_{v=0} = 0 \qquad \text{for } 1 \leqslant k \leqslant 2(n-1)$$

and hence (by inspection of the partial derivatives derived above)

$$\left.\frac{\partial^k |B|}{\partial \omega^k}\right|_{v=0} = 0 \qquad \text{for } 1 \leqslant k \leqslant 2(n-1)$$

From equation (41) it is readily seen that for $v = 0$, $\omega = \omega_0$ and hence

$$\left.\frac{\partial^k |B|}{\partial \omega^k}\right|_{\omega=\omega_0} = 0 \qquad \text{for } 1 \leqslant k \leqslant 2(n-1).$$

Hence the function $|B|$ defines a band pass filter which has $(n-1)$th order maximal flatness at $\omega = \omega_0$.

4.6. Poles of Transfer Function of Band Pass Filter

The poles of the required band pass function $B(S)$ are the poles of $B(S)\,B(-S)$ which are situated in the left half of the complex plane. From equation (32)

$$B(S)\,B(-S) = \frac{1}{1 + (-1)^n \left[\frac{\omega_0}{b}\left(\frac{S}{\omega_0} + \frac{\omega_0}{S}\right)\right]^{2n}}$$

$$= \frac{(bS)^{2n}}{(-1)^n\,[S^2 + \omega_0^2]^{2n} + 1} \tag{42}$$

$$B(S) = \frac{(bS)^n}{\text{Product of linear factors corresponding to poles of } B(S)\,B(-S) \text{ in left half of the complex plane}} \tag{43}$$

To find the poles of $B(S)\,B(-S)$ it is necessary to find the roots of the equation

$$(-1)^n\,(S^2 + \omega_0^2)^{2n} + 1 = 0 \tag{44}$$

Determination of the roots of equation (44) is equivalent to determination of the poles of $P(Z)\,P(-Z)$ (as given in equation (31)) in the S plane, where Z and S are related as indicated in equation (30).

From equation (30) we may write

$$S = \frac{Z}{2} \pm \sqrt{\left(\frac{Z}{2}\right)^2 - \omega_0^2} \tag{45}$$

Putting $S = Z$ in equations (23), (24) and (25) enables the poles of $P(Z)\,P(-Z)$ in the Z plane to be written down.

If we put $\qquad Z = b \operatorname{cis} \theta$

then for n even $\qquad \theta = \frac{\pi}{2n}(2N + 1) \qquad$ (See equation (24))

and for n odd $\qquad \varphi = \frac{\pi N}{n} \qquad$ (See equation (25))

where N is an integer.

Substituting $Z = b \operatorname{cis} \theta$ in equation (45) we obtain

$$S = \frac{b}{2}\cos\theta \pm A\cos\frac{\varphi}{2} + j\left(\frac{b}{2}\sin\theta \pm A\sin\frac{\varphi}{2}\right) \tag{46}$$

(Equation (45) represents two solutions for S obtained by taking the "+" signs together and then the "−" signs together.)

where
$$A = \sqrt[4]{\omega_0^4 + \left(\frac{b}{2}\right)^4 - \frac{(b\omega_0)^2}{2}\cos 2\theta}$$

$$\sin\varphi = \frac{\left(\frac{b}{2}\right)^2 \sin 2\theta}{A^2}$$

$$\cos\varphi = -\frac{\omega_0^2 - \left(\frac{b}{2}\right)^2 \cos 2\theta}{A^2}$$

$$\tan\varphi = -\frac{\sin 2\theta}{\left(\frac{2\omega_0}{b}\right)^2 - \cos 2\theta}$$

For each value of N (or θ) there are two solutions for S. As N goes from 0 to $2n - 1$ there will be $4n$ solutions. For $N \geqslant 2n$ the same solutions as given for $0 \leqslant N \leqslant 2n - 1$ are repeated. Hence equation (44) has $4n$ independent roots. In other words there will be $2n$ poles in each half plane or n conjugate pairs of poles in each half plane.

The following table shows the quadrant relation between θ, φ and $\varphi/2$.

	QUADRANT			
θ	1	2	3	4
φ	2	3	2	3
$\varphi/2$	1	2	1	2

If the roots of equation (44) are designated by $S_{01}, S_{02}, S_{11}, S_{12}, S_{21}, S_{22} \ldots S_{N1}, S_{N2} \ldots$ (S_{01}, S_{02} are the pair of roots for $N = 0$ etc.) where $0 \leqslant N \leqslant 2n - 1$ then equation (44) may be written as

$$(S - S_{01})(S - S_{02}) \ldots (S - S_{N1})(S - S_{N2}) \ldots = 0 \tag{47}$$

where $0 \leqslant N \leqslant 2n - 1$

Equation (47) may be expressed in terms of the product of real quadratic factors rather than as a product of complex linear factors. Using equation (46) a particular pair of roots designated by S_N (equivalent to the pair of roots S_{N1} and S_{N2}) may be written as

$$S_N = (u_N \pm v_N) + j(x_N \pm y_N) \tag{48}$$

$$\text{where } u_N = \frac{b}{2}\cos\theta$$

$$v_N = A\cos\frac{\varphi}{2}$$

$$x_N = \frac{b}{2}\sin\theta$$

$$y_N = A\sin\frac{\varphi}{2}$$

For n EVEN the following table applies:

N	0	1	2		N		$2n - 3$	$2n - 2$	$2n - 1$
θ	$\frac{\pi}{2n}$	$\frac{3\pi}{2n}$	$\frac{5\pi}{2n}$		$\frac{\pi}{2n}(2N + 1)$		$2\pi - \frac{5\pi}{2n}$	$2\pi - \frac{3\pi}{2n}$	$2\pi - \frac{\pi}{2n}$

From the above table it is obvious that

$$S_{(2n-1)-N} = (u_N \pm v_N) - j(x_N \pm y_N) \tag{49}$$

The roots given by equation (49) are conjugate to those given by equation (48). The product of the four roots given by these equations is

$$\{[S - (u_N + v_N)]^2 + (x_N + y_N)^2\}\{[S - (u_N - v_N)]^2 + (x_N - y_N)^2\}$$

If we take products as indicated in the above expression for the range $n/2 \leqslant N \leqslant n - 1$ (i.e. we consider all poles of $B(S)\,B(-S)$ lying in the left half plane) the transfer function $B(S)$ may readily be obtained.

It is convenient to replace N by another integer M where

$$M = N - \frac{n}{2}$$

$$\text{for } N = \frac{n}{2}, \qquad M = 0$$

$$\text{and for } N = n - 1, \qquad M = \frac{n}{2} - 1$$

$$\theta = \frac{\pi}{2n}(2N + 1) \quad = \frac{\pi}{2n}(2M + 1) + \frac{\pi}{2}$$

$$\text{Putting } \alpha = \frac{\pi}{2n}(2M + 1)$$

$$\text{we obtain } \sin\theta = \cos\alpha$$

$$\cos\theta = -\sin\alpha$$

$$\sin 2\theta = -\sin 2\alpha$$

$$\cos 2\theta = -\cos 2\alpha$$

Hence from equation (43)

$$B(S) = \frac{(bS)^n}{\prod_{M=0}^{n/2-1} \{(S + u_M + v_M)^2 + (x_M + y_M)^2\}\{(S + u_M - v_M)^2 + (x_M - y_M)^2\}} \quad \text{for } n \text{ even} \tag{50}$$

where

$$\alpha = \frac{\pi}{2n}(2M + 1)$$

$$u_M = \frac{b}{2} \sin \alpha$$

$$v_M = - A \cos \frac{\varphi}{2}$$

$$x_M = \frac{b}{2} \cos \alpha$$

$$y_M = A \sin \frac{\varphi}{2}$$

$$A = \sqrt[4]{\omega_0^4 + \left(\frac{b}{2}\right)^4 + \frac{(b\omega_0)^2}{2} \cos 2\alpha}$$

$$\sin \varphi = - \frac{\left(\frac{b}{2}\right)^2 \sin 2\alpha}{A^2}$$

$$\cos \varphi = - \frac{\omega_0^2 + \left(\frac{b}{2}\right)^2 \cos 2\alpha}{A^2}$$

$$\tan \varphi = \frac{\sin 2\alpha}{\left(\frac{2\omega_0}{b}\right)^2 + \cos 2\alpha}$$

For n ODD one pole of $P(Z)$ (refer to equation (25)) always occurs on the negative real axis in the Z plane and one pole of $P(-Z)$ always occurs on the positive real axis in the Z plane. The pole of $P(Z)$ on the negative real axis gives rise to a pair of conjugate poles of $B(S)$ in the S plane. These particular poles may be determined by putting $N = n$ in equation (25) for which $\theta = \pi$, and hence from equation (46)

$$S_n = -\frac{b}{2} \pm j\sqrt{\omega_0^2 - \left(\frac{b}{2}\right)^2}$$

$$S_{n1} = -\frac{b}{2} + j\sqrt{\omega_0^2 - \left(\frac{b}{2}\right)^2}$$

$$S_{n2} = -\frac{b}{2} - j\sqrt{\omega_0^2 - \left(\frac{b}{2}\right)^2}$$

$$(S - S_{n1})(S - S_{n2}) = \left(S + \frac{b}{2} - j\sqrt{\omega_0^2 - \left(\frac{b}{2}\right)^2}\right)\left(S + \frac{b}{2} + j\sqrt{\omega_0^2 - \left(\frac{b}{2}\right)^2}\right)$$

$$= \left(S + \frac{b}{2}\right)^2 + \omega_0^2 - \left(\frac{b}{2}\right)^2$$

$$= S^2 + bS + \omega_0^2$$

The quadratic factors in the denominator of $B(S)$ corresponding to the other roots of equation (44) which give rise to poles in the left hand half plane may be determined in a similar manner to the case for n even by considering the range $\frac{n+1}{2} \leqslant N \leqslant n - 1$ for $n > 1$.

In this case define $M = N - \frac{n+1}{2}$

For $N = \frac{n+1}{2}$ $\qquad M = 0$

and for $N = n - 1$ $\qquad M = \frac{n-3}{2}$

Put $\alpha = \frac{\pi}{2n}(2M + 1)$ as was done in the case for n even.

It follows that

$$B(S) = \frac{(bS)^n}{(S^2 + bS + \omega_0^2) \prod_{M=0}^{(n-3)/2} \{(S + u_M + v_M)^2 + (x_M + y_M)^2\}\{(S + u_M - v_M)^2 + (x_M - y_M)^2\}} \tag{51}$$

for n odd

where u_M, v_M, x_M and y_M are as defined for n even.

Note that for $n = 1$, the first factor only, in the denominator of equation (51), is considered.

5. RESPONSE OF BAND PASS FILTER TO SUDDENLY APPLIED SINUSOIDAL VOLTAGE

5.1. General Case

The response of a band pass filter to a finite burst of sine waves can be conveniently derived, using the principle of superposition, from the response of the filter to a sine wave suddenly applied and maintained for infinite time.

An excitation voltage corresponding to a sine wave suddenly applied and maintained for infinite time if defined in equation (52).

$$e_1(t) = H(t) \sin (\omega_C t + \beta) \tag{52}$$

where $H(t)$ is the Heaviside Unit Step Function.

An excitation voltage $e_i(t)$ consisting of a finite burst of sine waves may be written as

$$e_i(t) = [H(t) - H(t - T)] \sin (\omega_C t + \beta) \tag{53}$$

where T is the duration of the burst of sine waves.

From equation (53)

$$e_i(t) = e_1(t) - e_2(t) \tag{54}$$

where $e_2(t) = H(t - T) \sin (\omega_C t + \beta)$ (55)

For a linear system the response of the band pass filter to an excitation $e_i(t)$ is simply the response of the filter to $e_1(t)$ minus the response to $e_2(t)$. Note that $e_2(t)$ represents a burst of sine waves starting at $t = T$. If the burst of sine waves consists of an integral number of cycles then $e_2(t) = e_1(t - T)$. In other words, in that case, $e_2(t)$ is an identical function to $e_1(t)$ except that it is displaced in time. For simplicity the response of the filter to $e_1(t)$ only will be considered and it is assumed for reasons given above that the response of the filter to $e_i(t)$ can be easily deduced.

Using Laplace transformations, we obtain from equation (52)

$$E_1(S) = \frac{S \sin \beta + \omega_c \cos \beta}{S^2 + \omega_c^2} \tag{56}$$

If e_0 is the output of the band pass filter, for an input excitation e_1, then

$$E_0(S) = B(S)E_1(S)$$
$$= \frac{S \sin \beta + \omega_c \cos \beta}{S^2 + \omega_c^2} B(S) \tag{57}$$

For the following analysis it will be assumed that the sine wave burst starts from zero at $t = 0$, and hence $\beta = 0$.

In that case equation (57) simplifies to

$$E_0(S) = \frac{\omega_c}{S^2 + \omega_c^2} B(S) \tag{58}$$

Substituting for $B(S)$ from equations (50) and (51) we obtain

$$E_0(S) = \frac{\omega_C (bS)^n}{(S^2 + \omega_c^2) \prod_{M=0}^{n/2-1} \{(S + u_M + v_M)^2 + (x_M + y_M)^2\}\{(S + u_M - v_M)^2 + (x_M - y_M)^2\}} \tag{59}$$

for n even.

$$E_0(S) = \frac{\omega_C (bS)^n}{(S^2 + \omega_c^2)(S^2 + bS + \omega_0^2) \prod_{M=0}^{(n-3)/2} \{(S + u_M + v_M)^2 + (x_M + y_M)^2\}\{(S + u_M - v_M)^2 + (x_M - y_M)^2\}}$$

for n odd. (60)

It is obvious that E_0S can be broken up into partial fractions of the form

$$E_0S = \frac{b_{11}(S + a_{11}) + b_{12}a_{12}}{(S + a_{11})^2 + a_{12}^2} + \frac{b_{21}(S + a_{21}) + b_{22}a_{22}}{(S + a_{21})^2 + a_{22}^2} + \ldots \qquad (61)$$

which transforms into

$$e_0(t) = e^{-a_{11}t}(b_{11}\cos a_{12}t + b_{12}\sin a_{12}t) + e^{-a_{21}t}(b_{21}\cos a_{22}t + b_{22}\sin a_{22}t) + \ldots \qquad (62)$$

For $n > 1$, it is obvious that the determination of b_{11}, b_{12} etc., is an extremely laborious process, particularly when it is desired to retain the bandwidth b and the centre frequency ω_0 as variables.

Consider the simple case for $n = 1$

Assume that the frequency of the sinusoidal signal is made equal to the centre frequency of the band pass filter, i.e. $\omega_c = \omega_0$.

$$E_0(S) = \frac{b\omega_0 S}{(S^2 + \omega_0^2)(S^2 + bS + \omega_0^2)} \qquad (63)$$

$$E_0(S) = \omega_0\left[\frac{1}{S^2 + \omega_0^2} - \frac{1}{\left(S + \frac{b}{2}\right)^2 + \omega_0^2 - \left(\frac{b}{2}\right)^2}\right]$$

$$e_0(t) = H(t)\left[\sin\omega_0 t - \frac{\omega_0}{\sqrt{\omega_0^2 - \left(\frac{b}{2}\right)^2}}\, e^{-\frac{1}{2}bt}\sin t\sqrt{\omega_0^2 - \left(\frac{b}{2}\right)^2}\right] \qquad (64)$$

If we put $\frac{b}{2} = k\omega_0$ then

$$e_0(t) = H(t)\left(\sin\omega_0 t - \frac{1}{\sqrt{1 - k^2}}e^{-k\omega_0 t}\sin\omega_0 t\sqrt{1 - k^2}\right)$$

For k small (narrow band filter)

$$e_0(t) \simeq H(t)(1 - e^{-k\omega_0 t})\sin\omega_0 t \qquad (65)$$

It is of interest to note the form of the response for k large.

For k large the name "centre frequency" given to f_0 (see Sec. 4.4) is actually a misnomer. It is only in narrow band filters that the "centre frequency" lies approximately midway between the upper and lower half power frequencies. Substituting $\frac{b}{2} = k\omega_0$ in equations (36) and (37) we obtain

$$\omega_2 = 2\pi f_2 = (\sqrt{k^2 + 1} + k)\omega_0$$

$$\text{and } \omega_1 = 2\pi f_1 = (\sqrt{k^2 + 1} - k)\omega_0$$

where f_2 is the upper half power frequency and f_1 is the lower half power frequency.

Hence $$\frac{\omega_2 + \omega_1}{2} = \pi(f_2 + f_1) = \tfrac{1}{2}\omega_0\sqrt{k^2 + 1}$$

where $\frac{f_2 + f_1}{2}$ is the mid frequency of the pass band and is approximately equal to f_0 only for k small (narrow band filters).

Even for k large ω_1 is always greater than zero.

For $k < 1$

$$e_0(t) = H(t)\left(\sin\omega_0 t - \frac{1}{\sqrt{k^2 - 1}}e^{-k\omega_0 t}\sinh\omega_0 t\sqrt{k^2 - 1}\right) \qquad (66)$$

The damped sinusoid has disappeared from the second term, and a zero shift of the sinusoidal term occurs for small t. Note that the zero shift term is zero at $t = 0$ and approaches zero for t large.

Define $$q(t) = -\frac{1}{\sqrt{k^2 - 1}}e^{-k\omega_0 t}\sinh\omega_0 t\sqrt{k^2 - 1} \qquad (67)$$

For $q(t)$ to have a maximum value,

$$\frac{d[q(t)]}{dt} = 0$$

and for that case it follows that

$$t = \frac{ln(k + \sqrt{k^2 - 1})}{\omega_0\sqrt{k^2 - 1}} \tag{68}$$

$q(t)$ has been sketched as a function of $\omega_0 t$ in Fig. 8 for the special cases of $k = 1$ and $k = 10$.

Note that for infinite bandwidth (k infinite) there is no zero shift and the input signal is reproduced faithfully at the output of the filter.

For $n > 1$, an exact analysis is extremely involved. It will be shown in the next section that a considerable simplification is possible if the bandwidth of the band pass filter is small compared with the centre frequency (f_0).

5.2. Narrow Band Approximation for Band Pass Filter

Determination of the response of a band pass filter to a stepped sinusoid can be greatly simplified if a narrow band approximation for the filter is in order (Refs. 6 and 7). As mentioned earlier, the transfer function of the band pass filter may be derived from the transfer function of the low pass filter, using a suitable complex transformation. A filter having a maximally flat amplitude response will once again be considered.

Define the excitation for the band pass filter as

$$e_1(t) = H(t) \sin \omega_0 t$$

(a stepped sinusoid assumed to start at zero in this case).

Define the auxiliary function

$$f_1(t) = H(t)\, e^{j\omega_0 t}$$

Hence $$e_1(t) = Im\, H(t)e^{j\omega_0 t} = Im\, f_1(t)$$

Defining $F_1(S)$ as the Laplace Transform of $f_1(t)$ we may write

$$F_1(S) = \frac{1}{S - j\omega_0}$$

Similarly

$F(S) = F_1(S)B(S)$ where $B(S)$ is the transfer function of the band pass filter and $F(S)$ is the Laplace Transform of the response of the band pass filter.

The time response of the band pass filter is given by the Inverse Laplace Transform (Ref. 8, p. 126).

$$f(t) = \frac{1}{2\pi j} \lim_{R \to \infty} \int_{\gamma - jR}^{\gamma + jR} F(S)e^{St} dS \tag{69}$$

where the above integral is a contour integral and γ is chosen such that it is greater than the real part of all the singular points (i.e. poles) of $F(S)$.

Recapitulating, if $P(Z)$ is the transfer function of the low pass filter, then

$$P(Z)P(-Z) = \frac{1}{1 + (-1)^n\left(\frac{Z}{b}\right)^{2n}}$$

Substituting $Z = \omega_0\left(\frac{S}{\omega_0} + \frac{\omega_0}{S}\right)$ in the above we obtain $B(S)B(-S)$ where $B(S)$ is the transfer function of the band pass filter.

In Sec. 4.6 we found that $B(S)$ gives rise to n conjugate pairs of poles in the left hand half plane. Denoting the first pole by S_1 and its conjugate by $S_1{}^*$ etc. we may write

$$B(S) = \frac{(bS)^n}{(S - S_1)(S - S_1{}^*)(S - S_2)(S - S_2{}^*) \ldots (S - S_n)(S - S_n{}^*)} \tag{70}$$

Hence
$$F(S) = F_1(S)B(S)$$
$$= \frac{B(S)}{S - j\omega_0}$$
$$= \frac{(bS)^n}{(S - j\omega_0)(S - S_1)(S - S_1{}^*)(S - S_2)(S - S_2{}^*) \ldots (S - S_n)(S - S_n{}^*)}$$

It can be shown that the contour integration of equation (69) gives (Ref. 8, pp. 127 and 117)

$$f(t) = \sum \text{Residues of } F(S)e^{st} \tag{71}$$

$$= \sum \lim_{S \to S_k} (S - S_k)F(S)e^{st} \tag{72}$$

where S_k refers to the poles of $F(S)$ which are taken in turn in the summation. $F(S)$ contains $2n+1$ singular poles and hence gives rise to $2n+1$ residues.

For a narrow band filter $B(S)$ will give rise to a cluster of poles near $S=+j\omega_0$ and the conjugate set near $S=-j\omega_0$ (refer to Fig. 9(a)). The magnitude of $B(j\omega)$ will peak at $S=+j\omega_0$ and at $S=-j\omega_0$ as depicted in Fig. 9(b). Note that $\omega_0/2\pi$ is the centre frequency of the band pass filter and also the frequency of the cisoidal function $f_1(t)$.

From equation (71) it is therefore obvious that $f(t)$ is the summation of a number of residues due to poles which lie close to $S=+j\omega_0$ and $S=-j\omega_0$. The residues due to poles near $S=-j\omega_0$ will be considerably smaller than the residues due to poles near $S=+j\omega_0$ because of the pole at $S=j\omega_0$ (no conjugate pole at $S=-j\omega_0$). Hence if $F(S)$ is approximated in the region of $S=+j\omega_0$ then a corresponding approximate solution will be obtained for the Inverse Transform $f(t)$.

Consider the transformation

$$S = h + j\omega_0 \tag{73}$$

$$= j\omega_0\left(1 - j\frac{h}{\omega_0}\right)$$

In the region of interest ($S \approx j\omega_0$) the magnitude of $\frac{h}{\omega_0}$ must be much less than unity. In other words $\frac{|h|}{\omega_0} \ll 1$ in the region of interest.

To obtain a band pass function $B(S)$, having a maximally flat amplitude characteristic at $\omega = \omega_0$, from a low pass function $P(S)$ having a maximally flat amplitude characteristic at $\omega = 0$ (see Sec. 4.3) we write

$$B(S) = P(Z)$$

$$\text{where } Z = \omega_0\left(\frac{S}{\omega_0} + \frac{\omega_0}{S}\right)$$

Substituting for S from equation (73) we obtain

$$Z = \omega_0\left(\frac{h + j\omega_0}{\omega_0} + \frac{\omega_0}{h + j\omega_0}\right)$$

$$= \omega_0\left(\frac{h}{\omega_0} + j - \frac{j}{1 - j\frac{h}{\omega_0}}\right)$$

$$= \omega_0\left(\frac{h}{\omega_0} + j - \frac{\left(j - \frac{h}{\omega_0}\right)}{1 + \left(\frac{h}{\omega_0}\right)^2}\right)$$

$$\simeq 2h$$

assuming $\left(\frac{|h|}{\omega_0}\right)^2 \ll 1$

Hence from equation (73) we may write

$$Z \simeq 2(S - j\omega_0) \tag{74}$$

for a narrow band filter.

Now $P(S)P(-S) = P(S, b)P(-S, b) = \dfrac{1}{1 + (-1)^n\left(\frac{S}{b}\right)^{2n}}$

where $P(S, b)$ means that P is a function of both S and b. Up to the present time the function $P(S, b)$ has been written simply as $P(S)$.

$$P(Z, b)P(-Z, b) = \frac{1}{1 + (-1)^n\left(\frac{Z}{b}\right)^{2n}}$$

gives the transfer function of the band pass filter having bandwidth b (equal to the bandwidth of the low pass filter) where $Z = \omega_0\left(\frac{S}{\omega_0} + \frac{\omega_0}{S}\right) \simeq 2(S - j\omega_0)$ for a narrow band filter.

$$P(Z, 2b)P(-Z, 2b) = \frac{1}{1 + (-1)^n\left(\frac{Z}{2b}\right)^{2n}}$$

$$= \frac{1}{1 + (-1)^n\left(\frac{S - j\omega_0}{b}\right)^{2n}}$$

gives the transfer function of a band pass filter having twice the bandwidth of the low pass filter.

Hence if $P(S)$ is the transfer function of the low pass filter then $P(S - j\omega_0)$ is the transfer function of the band pass filter having twice the bandwidth of the low pass filter.

The significance of the result indicated in equation (74) may be readily assessed from the following table:

	Excitation		Filter Transfer Function	Response	
	Time Function	Laplace Transform of Time Function		Laplace Transform of Time Function	Time Function
Low Pass Filter	$H(t)$ (Unit step)	$\frac{1}{S}$	$P(S)$ (Bandwidth b)	$\frac{P(S)}{S}$	$f_L(t)$
Band Pass Filter	$H(t)e^{j\omega_0 t}$ (Stepped Cisoid)	$\frac{1}{S - j\omega_0}$	$P(S - j\omega_0)$ (Bandwidth $2b$)	$\frac{P(S - j\omega_0)}{S - j\omega_0}$	$e^{j\omega_0 t}f_L(t)$

$f_L(t)$ is the time response of the low pass filter to a unit step input.

By taking real and imaginary parts of the input and output time functions of the band pass filter, we obtain

$$\text{Input} = H(t) \sin \omega_0 t$$

$$\text{Output} = f_L(t) \sin \omega_0 t$$

Summarising this result we can say that the amplitude of the response of a band pass filter of centre frequency $\omega_0/2\pi$ and narrow bandwidth to a suddenly applied sinusoidal carrier, $H(t) \sin \omega_0 t$, is equal to the unit step response of the equivalent low pass filter having half the bandwidth of the band pass filter. By equivalent low pass filter is meant the filter from which the band pass filter was derived by the transformation which has been defined in equation (30). The responses of the low pass filter and the equivalent band pass filter are illustrated in Fig. 10.

5.3. Response of the Narrow Bandwidth Maximally Flat Band Pass Filter to a Suddenly Applied Sinusoidal Voltage

We consider the transfer function of the maximally flat low pass filter.

$$P(S)P(-S) = \frac{1}{1 + (-1)^n\left(\frac{S}{b}\right)^{2n}}$$

The poles of $P(S)$ are given by (refer to Sec. 4.1, equations (24) and (25)).

$$S = b \operatorname{cis} \theta$$

$$\text{where } \theta = \frac{\pi}{2n}(2N + 1) \qquad \text{for } n \text{ even}$$

$$\theta = \frac{\pi N}{n} \qquad \text{for } n \text{ odd}$$

The analysis of Sec. 5.3 reveals that the amplitude response of the band pass filter of narrow bandwidth to a suddenly applied sinusoid is given by the step response of the equivalent low pass filter.

Let $a(t)$ be the step response of the low pass filter of bandwidth b, and let $A(S)$ be its transform.

$$A(S) = \frac{P(S)}{S}$$

The time responses will now be calculated for $n = 1, 2, 3$ and 4.

$\underline{n = 1}$

$$P(S) = \frac{b}{S + b}$$

$$A(S) = \frac{P(S)}{S}$$

$$= \frac{b}{S(S+b)}$$

$$= \frac{1}{S} - \frac{1}{S+b}$$

$$a(t) = 1 - e^{-bt} \tag{75}$$

$\underline{n = 2}$

$$A(S) = \frac{b^2}{S\left\{\left(S + \frac{b}{\sqrt{2}}\right)^2 + \left(\frac{b}{\sqrt{2}}\right)^2\right\}}$$

$$= \frac{1}{S} - \frac{\left(S + \frac{b}{\sqrt{2}}\right) + \frac{b}{\sqrt{2}}}{\left(S + \frac{b}{\sqrt{2}}\right)^2 + \left(\frac{b}{\sqrt{2}}\right)^2}$$

$$a(t) = 1 - e^{-\frac{b}{\sqrt{2}}t}\left(\cos\frac{b}{\sqrt{2}}t + \sin\frac{b}{\sqrt{2}}t\right)$$

$$= 1 - \sqrt{2}\, e^{-\frac{b}{\sqrt{2}}t}\sin\left(\frac{\pi}{4} + \frac{b}{\sqrt{2}}t\right) \tag{76}$$

$\underline{n = 3}$

$$A(S) = \frac{b^3}{S(S+b)\left[\left(S + \frac{b}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}b\right)^2\right]}$$

$$= \frac{1}{S} - \frac{1}{S+b} - \frac{b}{\left(S + \frac{b}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}b\right)^2}$$

$$a(t) = 1 - e^{-bt} - \frac{2}{\sqrt{3}} e^{-\frac{b}{2}t}\sin\frac{\sqrt{3}}{2}bt \tag{77}$$

$\underline{n = 4}$

$$A(S) = \frac{b^4}{S\left[\left(S + b\cos\frac{3\pi}{8}\right)^2 + \left(b\sin\frac{3\pi}{8}\right)^2\right]\left[\left(S + b\cos\frac{\pi}{8}\right)^2 + \left(b\sin\frac{\pi}{8}\right)^2\right]}$$

$$= \frac{1}{S} + \frac{\cos\frac{3\pi}{8}\left(S + b\cos\frac{3\pi}{8}\right) + \frac{b}{2}\cos\frac{3\pi}{4}}{\left(\cos\frac{\pi}{8} - \cos\frac{3\pi}{8}\right)\left[\left(S + b\cos\frac{3\pi}{8}\right)^2 + \left(b\sin\frac{3\pi}{8}\right)^2\right]}$$

$$\frac{\cos\frac{\pi}{8}\left(S + b\cos\frac{\pi}{8}\right) + \frac{b}{2}\cos\frac{\pi}{4}}{\left(\cos\frac{\pi}{8} - \cos\frac{3\pi}{8}\right)\left[\left(S + b\cos\frac{\pi}{8}\right)^2 + \left(b\sin\frac{\pi}{8}\right)^2\right]}$$

$$a(t) = 1 + e^{-\left[b\cos\frac{3\pi}{8}\right]t}\sin\left(\frac{3\pi}{4} + \left[b\sin\frac{3\pi}{8}\right]t\right) - \frac{1}{\sqrt{2}-1}e^{-\left[b\cos\frac{\pi}{8}\right]t}\sin\left(\frac{\pi}{4} + \left[b\sin\frac{\pi}{8}\right]t\right)$$

$$a(t) = 1 + e^{-tb\cos\frac{3\pi}{8}} \sin\left(\frac{\pi}{4} - tb\sin\frac{3\pi}{8}\right) - \frac{e^{-tb\cos\frac{\pi}{8}}}{\sqrt{2}-1} \sin\left(\frac{\pi}{4} + tb\sin\frac{\pi}{8}\right) \qquad (78)$$

Note that in each case the bandwidth of the band pass filter is $2b$.

Put bandwidth of bandpass filter equal to $k\omega_0$

$$b = \tfrac{1}{2}k\omega_0$$

Summarizing these results

$n = 1 \quad a(t) = 1 - e^{-0.5k\omega_0 t}$

$n = 2 \quad a(t) = 1 - 1{\cdot}414e^{-0.354k\omega_0 t} \sin\left(\frac{\pi}{4} + 0{\cdot}354k\omega_0 t\right)$

$n = 3 \quad a(t) = 1 - e^{-0.5k\omega_0 t} - 1{\cdot}155e^{-0.25k\omega_0 t} \sin(0{\cdot}433k\omega_0 t)$

$n = 4 \quad a(t) = 1 + e^{-0.191k\omega_0 t} \sin\left(\frac{\pi}{4} - 0{\cdot}462k\omega_0 t\right) - 2{\cdot}414e^{-0.433k\omega_0 t} \sin\left(\frac{\pi}{4} + 0{\cdot}191k\omega_0 t\right)$

The amplitude responses given by the above set of equations have been plotted in Fig. 11.

5.4. Practical Application of Response Curves Derived

The curves of Fig. 11 reveal that overshoot occurs for $n > 1$ and becomes more pronounced as n is made larger. The rate of rise in amplitude, say between 10% and 90% of full amplitude, is not significantly different for the values of n chosen. However, as n is made larger, the initial rate of rise in amplitude becomes less and the responses appear to have an initial time delay associated with them. The table below clearly indicates these effects.

Amplitude Limits	$k\omega_0 t$			
	$n = 1$	$n = 2$	$n = 3$	$n = 4$
0% to 10%	0·20	1·06	2·00	3·10
0% to 90%	4·60	5·32	6·60	7·90
10% to 90%	4·40	4·26	4·60	4·80

(Note that $k\omega_0 t = (2\pi k)f_0 t$ where $f_0 t$ is the number of cycles which have elapsed.)

The dimensionless product $k\omega_0 t$ (filter bandwidth multiplied by time) is the generalized variable against which the amplitudes have been plotted. It is clear from the response equations that the rate of rise in amplitude with time, for a given filter configuration, is proportional to the filter bandwidth. Expressed analytically

$$\frac{da}{dt} = \frac{da}{d(k\omega_0 t)} \cdot \frac{d(k\omega_0 t)}{dt}$$

$$= k\omega_0 \frac{da}{d(k\omega_0 t)}$$

$= \text{bandwidth in rads/sec} \times \text{gradient of graph.}$

The solution to any practical problem will in general be a compromise one. If the carrier frequency, f_0, is liable to vary between limits (as was the case for the A.R.L. Flight Memory Ground Station Equipment which was designed to accommodate $\pm 3\%$ wire speed variations between record and playback), then the bandwidth of the filter must be made large enough to accommodate the frequency changes. The proximity (on the frequency scale) of adjacent data channels will determine the amount of attenuation required outside the pass band of the filter. On the other hand the requirement that the rise in amplitude of a carrier suddenly applied to the filter be as rapid as possible implies that the filter bandwidth should be as large as possible, a requirement which conflicts with the demands on the steady state response; hence the need for a compromise solution.

As an example of a practical use for the graphs, the filtering requirement of the A.R.L Flight Memory Project will be considered. In that system flight data was recorded in the form of a burst of 3500 cycle/sec. sine waves. The duration of the burst was made proportional to the parameter being sampled. Cockpit speech and data were mixed to form a composite signal. A filter was required on playback to separate the data from the speech. To provide adequate separation between speech and data, it was essential that the response of the data band pass filter at 2500 cycles/second be down at least 20*db* on the response at 3500 cycles/second. In order to measure the value of the sampled parameter it was necessary to count the number of sine waves occurring during the particular burst. Full wave rectification (frequency doubling) of the output of the filter enabled half cycle resolution to be obtained. The peak noise level appearing at the output of the filter (essentially a restricted band noise centred on the centre frequency of the filter) could be taken to be about 20 *db* below the signal level. A filter having a bandwidth of about 700 cycles/second was chosen ($k = 0 \cdot 2$) and the filter configuration was similar to a maximally flat filter having $n = 2$.

For $$f = 2500 \text{ c/s}, \frac{f}{f_0} = \frac{2500}{3500} = 0 \cdot 71.$$

For the case $n = 2$ the frequency response is down $21 \cdot 5$ *db* (refer to Fig. 7), which is adequate for the design requirement.

Using the graph of Fig. 11 for the case of $n = 2$ we can predict the transient response of the filter to a suddenly applied sinusoidal voltage. Consider firstly the number of cycles of the 3500 cycle/second carrier which have lapsed before the response reaches 90% amplitude.

For $a(t) = 0 \cdot 9$ $(n = 2)$ $k\omega_0 t = 5 \cdot 40$

$$f_0 t = \frac{5 \cdot 40}{2\pi k} = \frac{5 \cdot 40}{0 \cdot 4\pi} = 4 \cdot 3 \text{ cycles.}$$

In other words about $4\frac{1}{2}$ cycles lapse before the response reaches 90% amplitude.

Counting of the number of cycles can be achieved automatically by counting the number of times the signal exceeds a certain level. If the sampling level is made too high noise and drops in signal amplitude will introduce errors in the counting process. Similarly if the counting level is made too low noise pulses between bursts of signal may be counted. A suitable counting level is about the 50% amplitude level, where the rate of amplitude rise approaches a maximum. In the presence of a $\pm 10\%$ noise signal counting may commence anywhere between 40% and 60% amplitude levels. From the graph of Fig. 11, for $n = 2$, it is observed that $k\omega_0 t$ changes by $0 \cdot 92$ over this amplitude range.

$$f_0 t = \frac{0 \cdot 92}{2\pi k} = 0 \cdot 73 \text{ cycle } (k = 0 \cdot 2).$$

Hence the error in count at the start of the burst may be about $\pm\frac{1}{2}$ cycle for this level of noise. It is to be emphasised that the loss of a few cycles at the start does not lead to inaccuracies as the difference in counts will be constant and can be taken into account.

The principle of superposition may be used to deduce the manner in which the filter will respond when the signal is switched off. The portion of the response after the sinusoidal signal is switched off is simply the subtraction of the response of the filter to a sine wave suddenly applied at that time. If overshoot occurs at switch on then a secondary lobe will appear during the "ringing on" period after the excitation is removed. The effect is depicted in Fig. 12. The uncertainty in count at the trailing edge of the filtered pulse because of the finite decay time and the presence of noise will be essentially the same as at the leading edge. Hence a total error of ± 1 count is to be expected.

6. CONCLUSION

The response of a band pass filter, having a maximally flat amplitude characteristic, has been calculated for the case of an excitation voltage consisting of a sine wave which is abruptly switched on and off. Laplace transform methods have been used to enable the response of a physically realizable filter (maximally flat amplitude type) to be calculated. It has been shown that great simplifications result in the calculations if a suitable low pass to band pass transformation is used and provided that the pass-band of the band pass filter is sufficiently narrow.

In that case the amplitude of the response of the band pass filter to a suddenly applied sinusoidal carrier is equivalent to the unit step response of the equivalent low pass filter having half the bandwidth of the band pass filter.

The Fourier Integral provides a useful indication of the frequency components present in the excitation voltage and hence indicates the main frequencies to be passed by a band pass filter if reasonable fidelity is to be obtained. The response of an idealized band pass filter having constant gain and zero phase shift within the pass band and infinite attenuation outside the band can be readily computed using the Fourier Integral approach. However such a filter is not physically realizable and is really of academic interest only. In order to obtain the time response of a physically realizable filter using Fourier methods it would be necessary to specify both amplitude and phase characteristics of the band pass filter and perform suitable integration to find the time response. Such a method can be shown to be equivalent to the method using the Laplace transform. Hence there appears to be little merit in the Fourier Integral approach for obtaining the time response of physically realizable filters.

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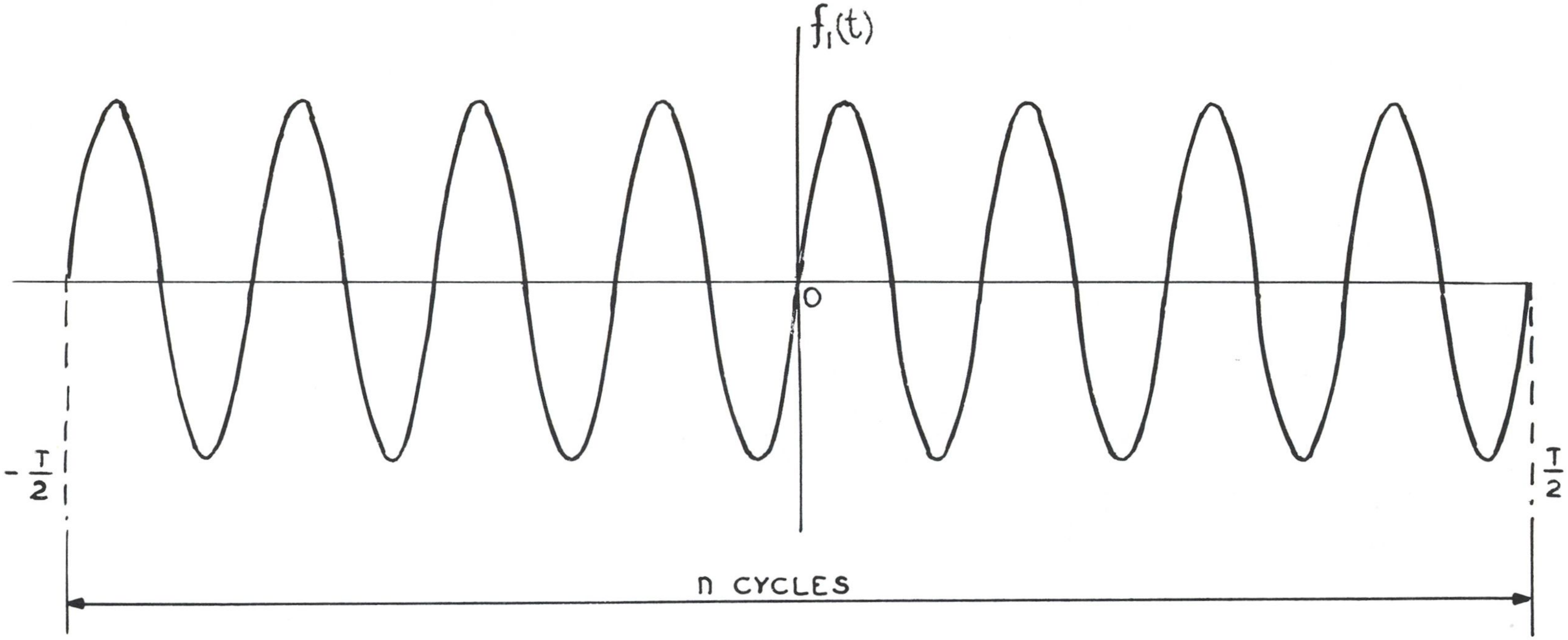
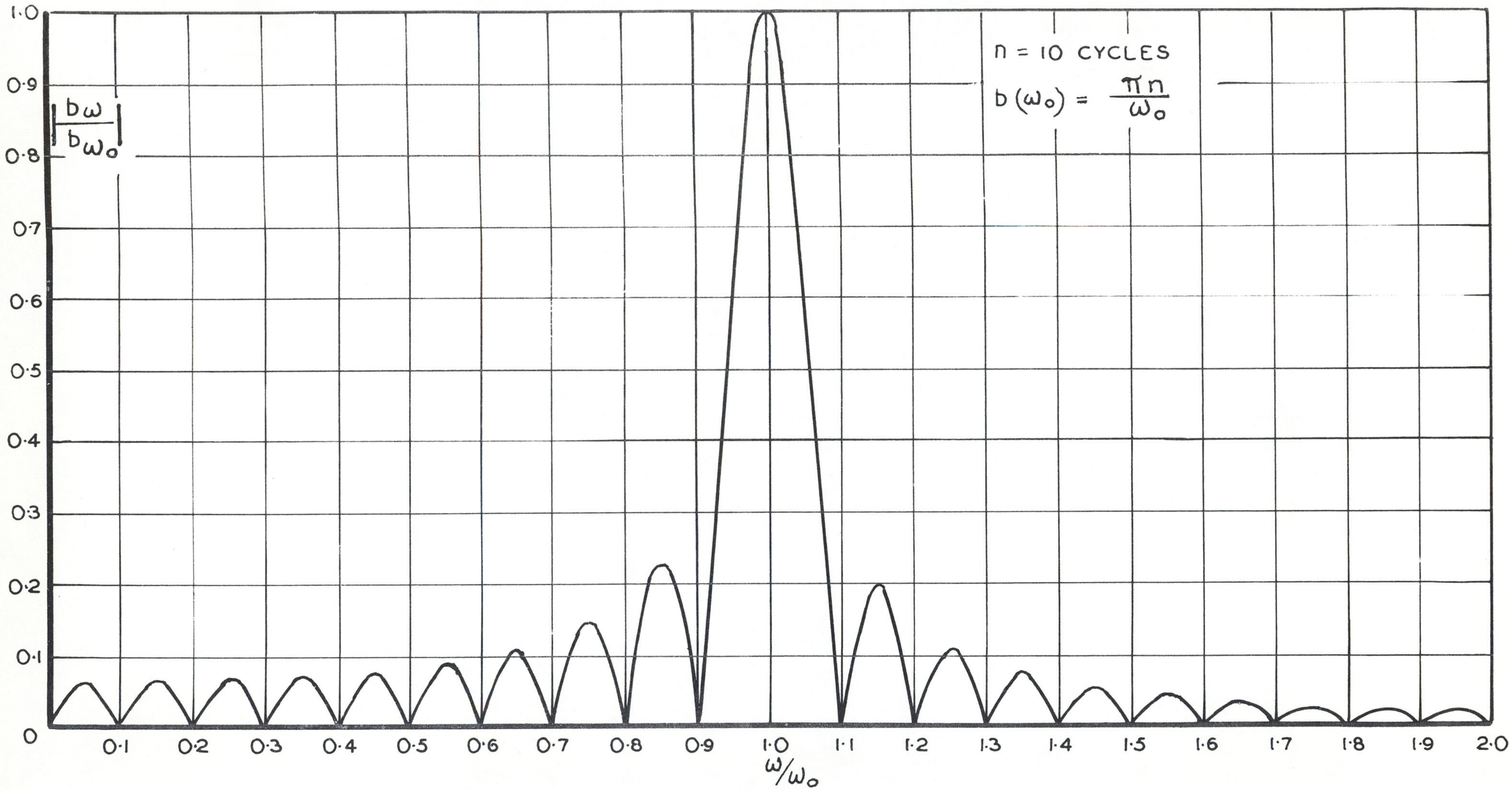


SINE WAVE BURST DEFINED BY $f_1(t) = \text{SIN}\ \frac{\omega_0 t}{T}$ IN RANGE $-\frac{T}{2} < t < \frac{T}{2}$

INST. NOTE 70
FIG. 1



FREQUENCY SPECTRUM OF A BURST OF SINE WAVES

INST. NOTE 70
FIG. 2

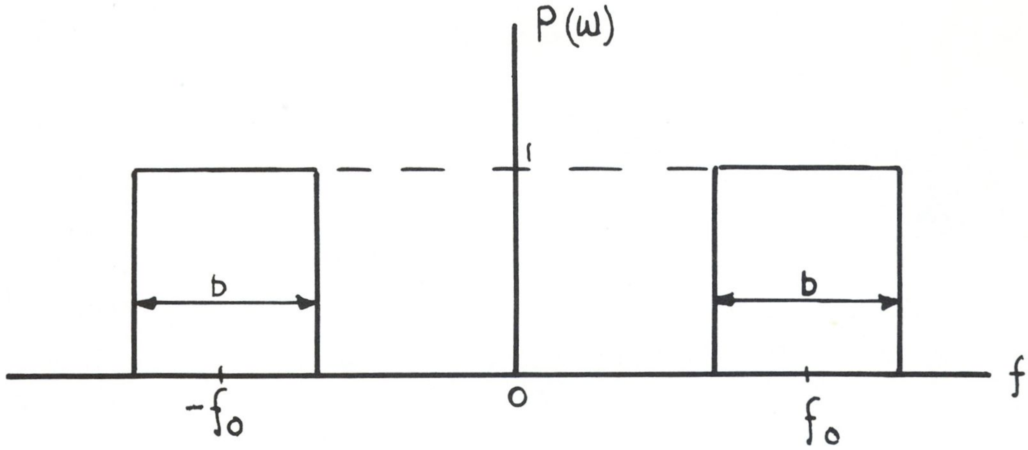


FIG. 3 FREQUENCY RESPONSE OF IDEAL BAND PASS FILTER

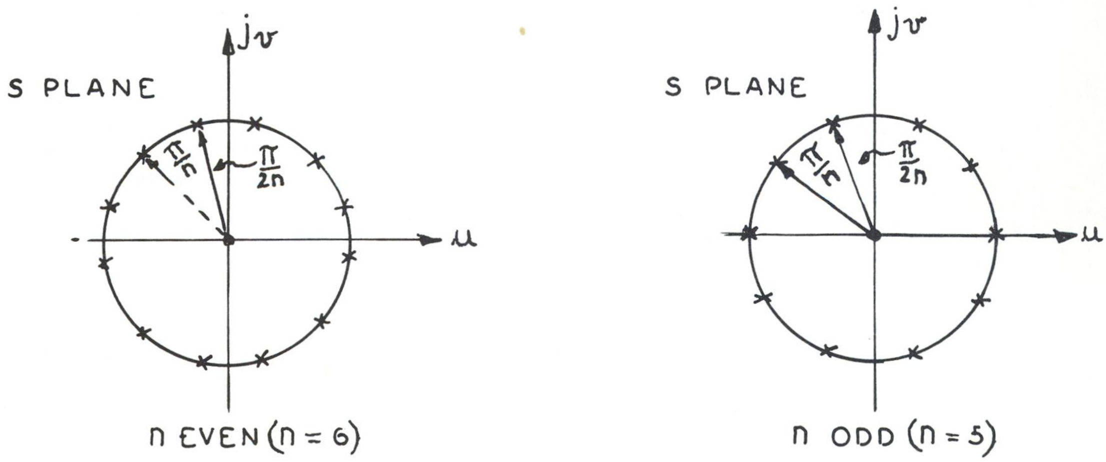


FIG. 4 POLE CONFIGURATION OF P(S) P(-S)

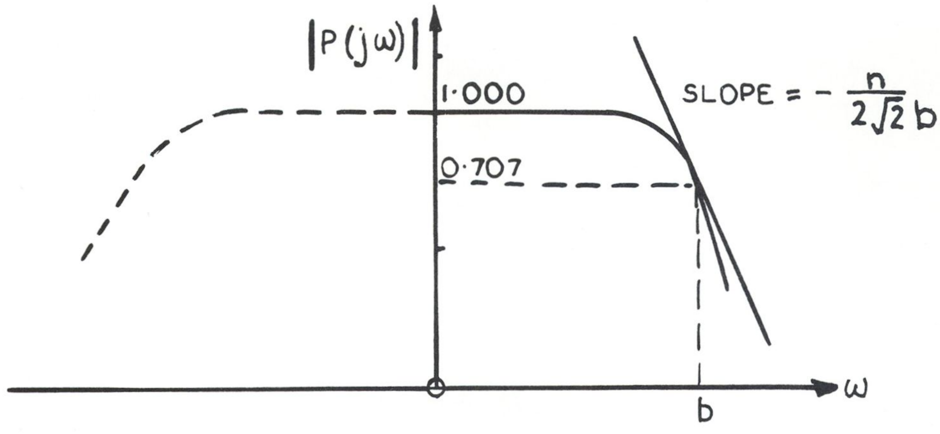


FIG. 5 FREQUENCY RESPONSE OF LOW PASS FILTER WITH MAXIMALLY FLAT AMPLITUDE RESPONSE

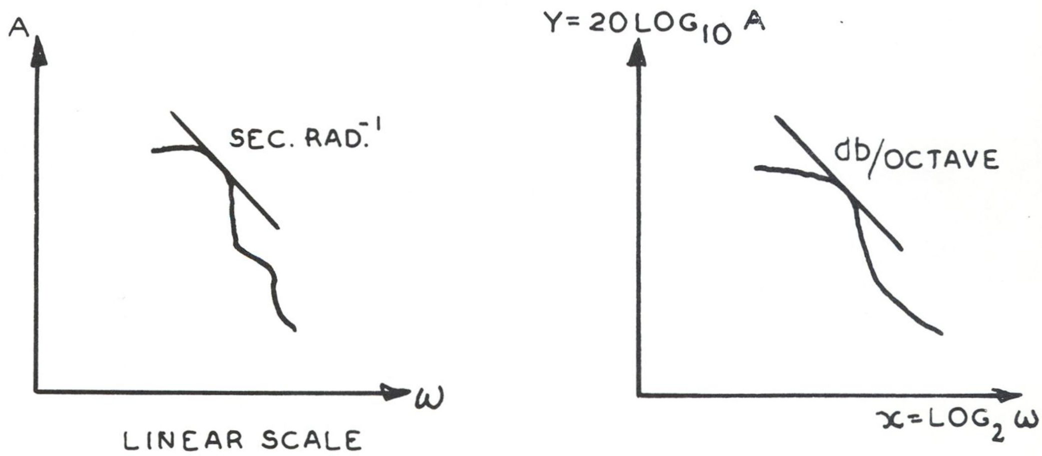
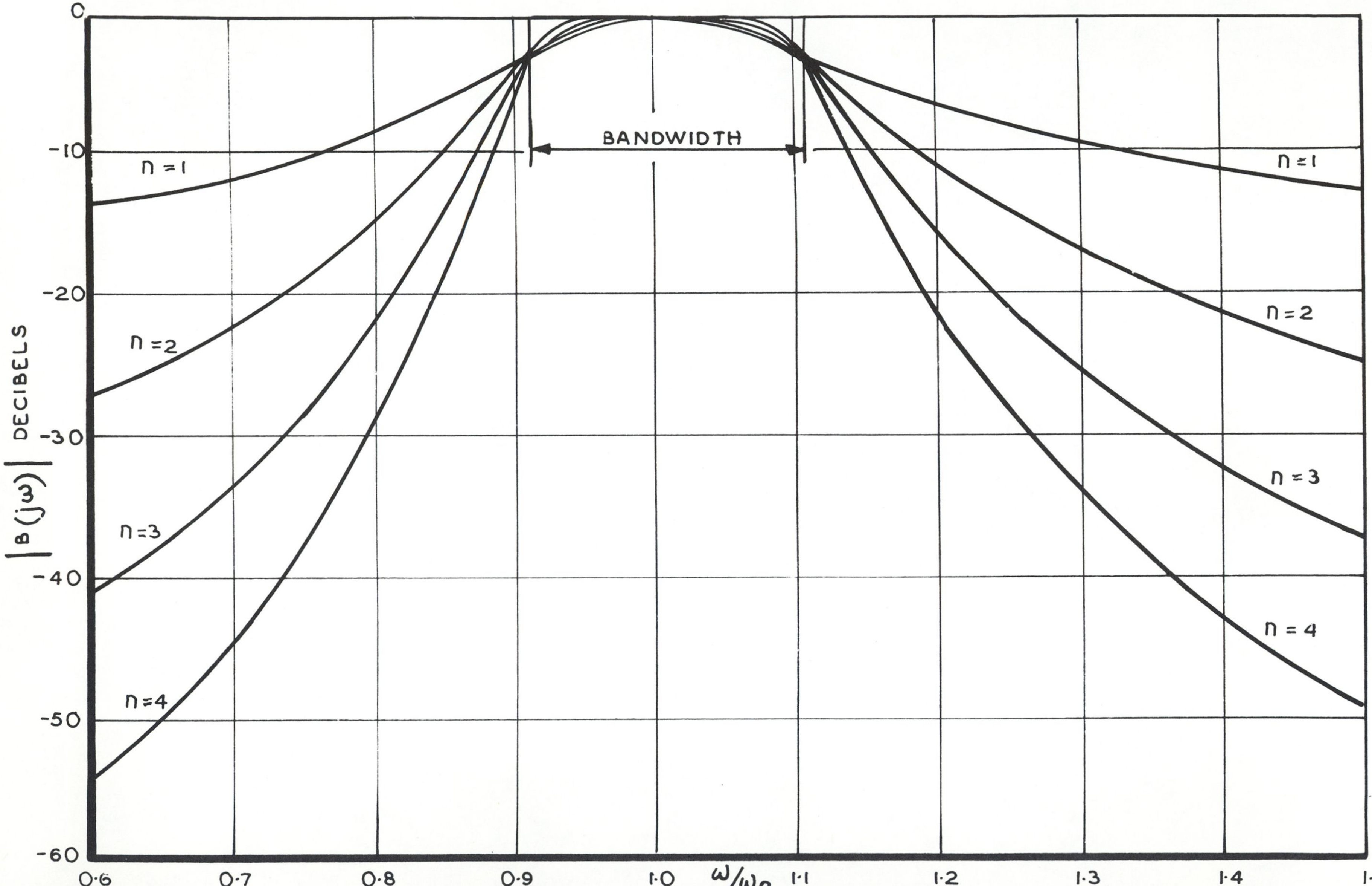
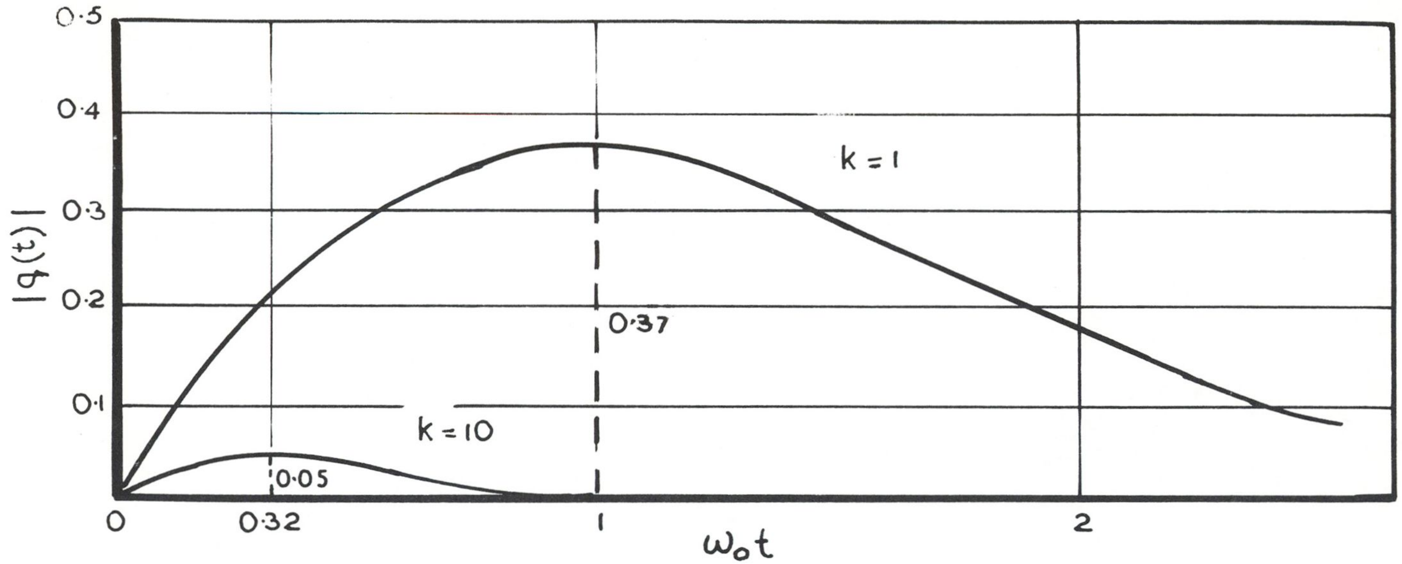


FIG. 6 SLOPE OF FREQUENCY RESPONSE CHARACTERISTIC IN LOGARITHMIC UNITS



FREQUENCY RESPONSES OF MAXIMALLY FLAT BAND PASS FILTERS FOR $\frac{b}{\omega_0} = 0{\cdot}2$

INST. NOTE 70
FIG. 7



PLOT OF $|q(t)|$ FOR $k=1$ AND $k=10$

INST. NOTE 70
FIG.8

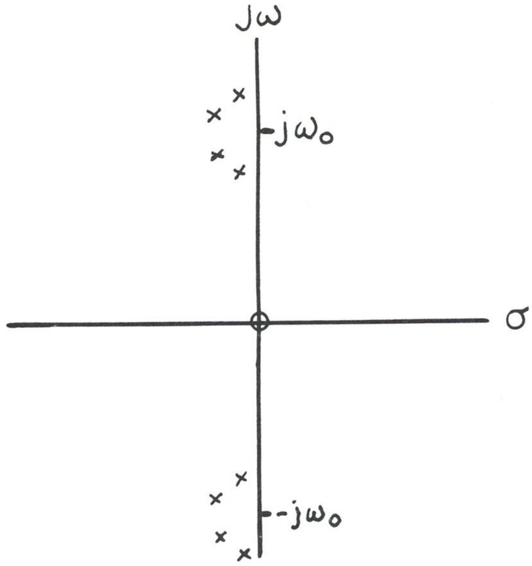


FIG. 9(a) POLE CONFIGURATION FOR BAND PASS FILTER

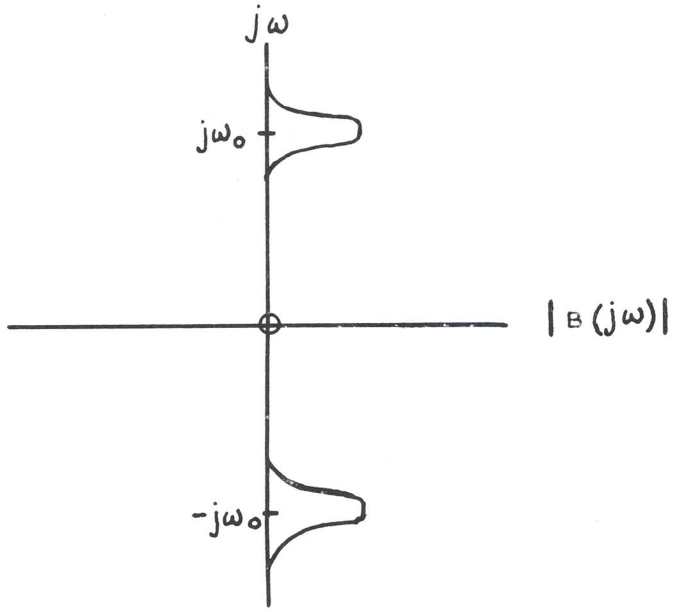
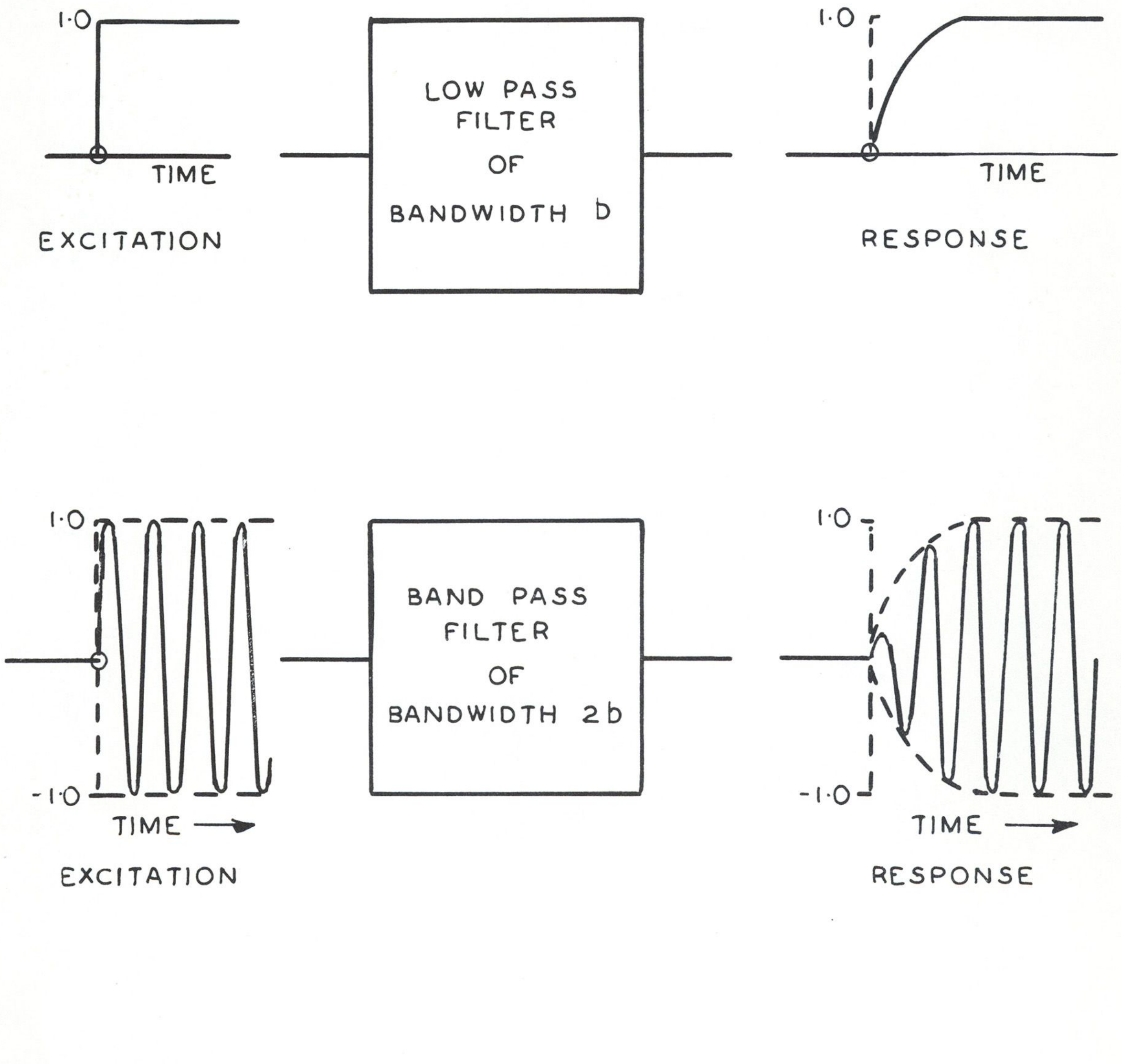
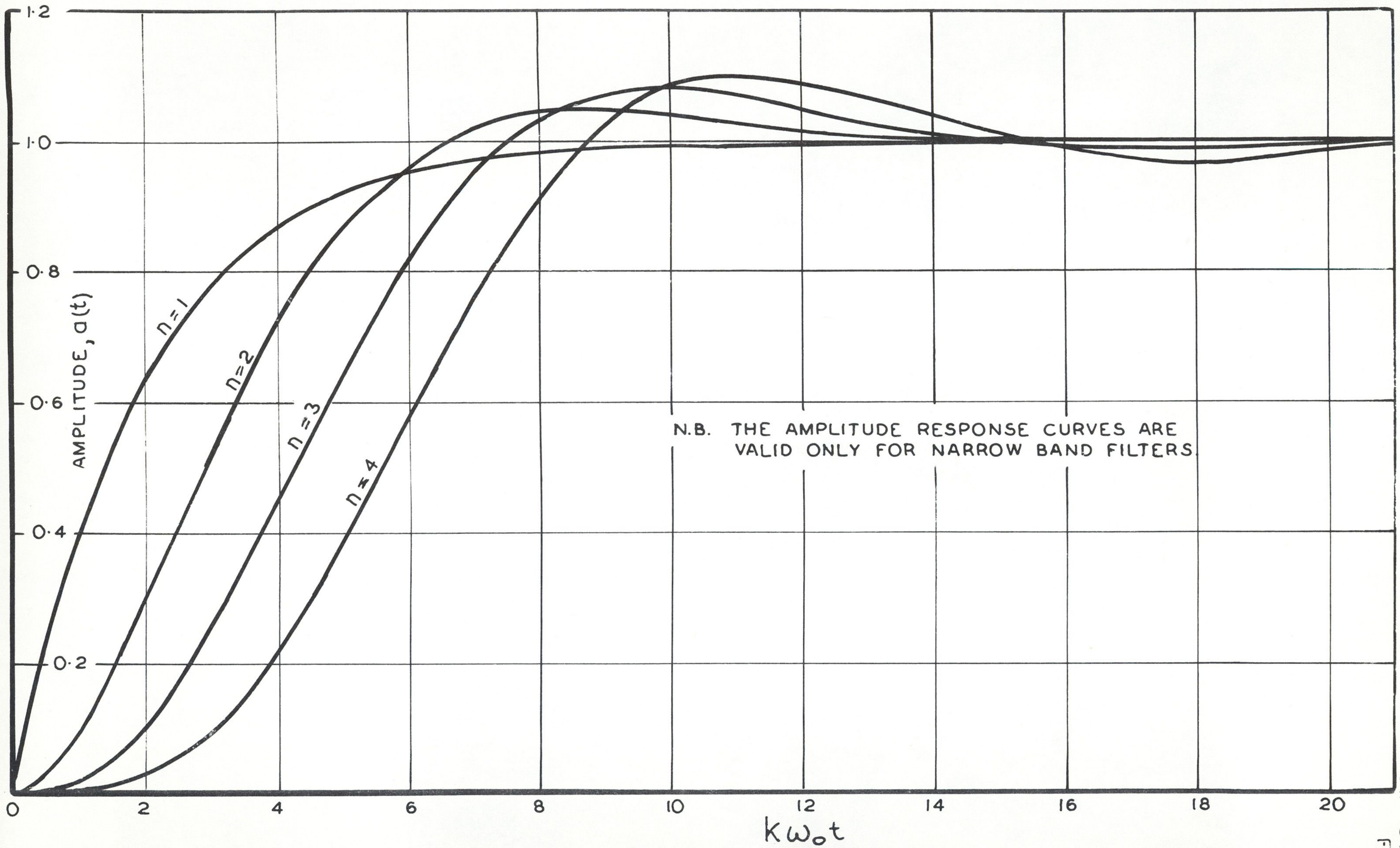


FIG. 9(b) AMPLITUDE RESPONSE OF BAND PASS FILTER



AMPLITUDE RESPONSE OF A BAND PASS FILTER IN TERMS OF THE RESPONSE OF THE EQUIVALENT LOW PASS FILTER



AMPLITUDE RESPONSES OF BAND PASS FILTERS HAVING MAXIMALLY FLAT AMPLITUDE CHARACTERISTICS

INST. NOTE 70
FIG. 11

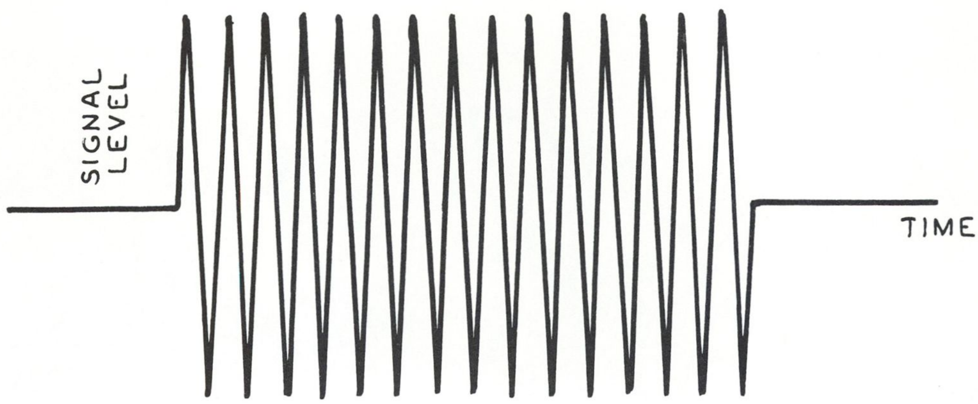


FIG. 12(a) BAND PASS FILTER EXCITATION

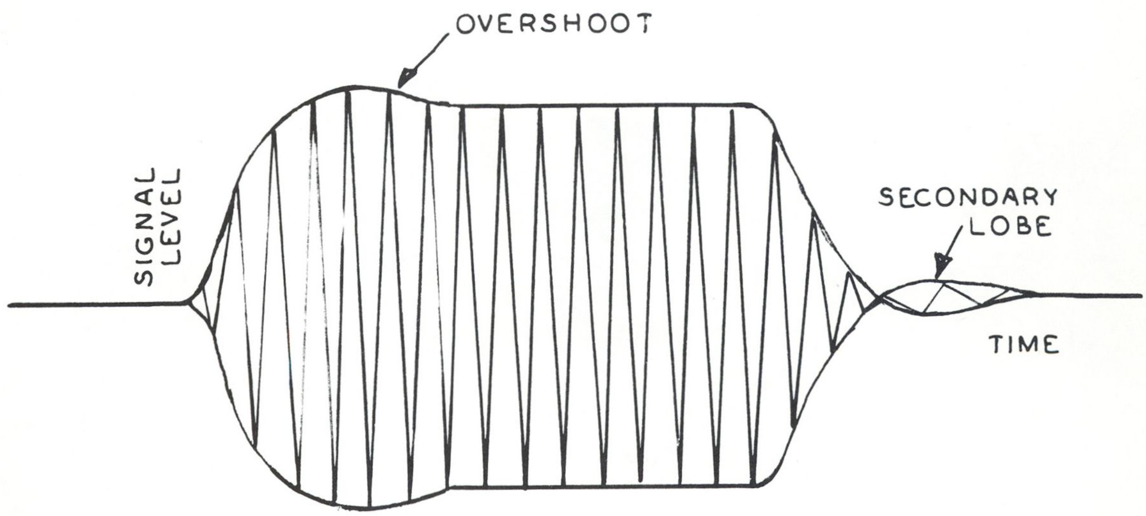


FIG. 12(b) TYPICAL BAND PASS FILTER RESPONSE